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WOVEN g-FRAMES IN HILBERT C^*-MODULES

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Abstract. Woven frames are motivated from distributed signal processing with potential applications in wireless sensor networks. g-frames provide more choices on analyzing functions from the frame expansion coefficients. The objective of this paper is to introduce woven g-frames in Hilbert C*-modules, and to develop its fundamental properties. In this investigation, we establish sufficient conditions under which two g-frames possess the weaving properties. We also investigate the sufficient conditions under which a family of g-frames possess weaving properties.

1. Introduction

Frames in Hilbert spaces were first proposed by Duffin and Schaeffer [10] in 1952 while studying the nonharmonic Fourier series. Frames can be viewed as more flexible substitutes of bases in Hilbert spaces. They are more flexible tools as linear independence between the frame elements are not required. In 1985, as the wavelet era began, Daubechies, Grossmann, and Meyer [9] reintroduced and developed the theory of frames in 1986. Due to its remarkable structure, the subject drew the attention of many mathematicians, physicists, and engineers because of its wide application in various well known fields like signal processing [4], coding and communications [19], image processing [5], sampling theory [11], numerical analysis, filter theory [3]. In recent years, it has emerged as an important tool in compressive sensing, data analysis, and in several other areas. The notion of woven frames in Hilbert space was introduced by Bemrose et al. [2], and more deeply investigated in [7, 8]. The concept of woven frames is partially motivated by preprocessing of Gabor frames, and has potential applications in wireless sensor networks that require distributed processing under different frames. In the past few years, several generalizations of frames in Hilbert space have been proposed, for example, fusion frames [6], pseudo-frames [16], etc. Sun [20] introduced the concept of g-frame or generalized frames in Hilbert spaces. Let $\mathcal{X}$ and $\mathcal{Y}$ be separable Hilbert spaces, and $\{\mathcal{Y}_i : i \in I\}$ be a sequence of closed subspaces of $\mathcal{Y}$. Let $\mathcal{L}(\mathcal{X}, \mathcal{Y}_i)$ be the collection of all bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}_i$.

Definition 1.1. [20] A sequence $\{\Lambda_j \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_j) \colon j \in \mathbb{J}\}$ is called a generalized frame, or simply a g-frame, for $\mathcal{X}$ with respect to $\{\mathcal{Y}_j : j \in J\}$ if there are two positive

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constants A and B such that

$$A\|f\|^2 \leq \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \leq B\|f\|^2, \ \forall f \in \mathcal{X}. \tag{1.1}$$

Woven frames are powerful tools in wireless sensor network. In Hilbert spaces, they are defined as follows:

Definition 1.2. [2] Let $[m] = \{1, 2, ..., m\}$. A family of frames $\{\phi_{ij}\}_{i\in I}$ for $j \in [m]$ for a Hilbert space H is said to be woven, if there are universal positive constants A and B such that for every partition $\{\sigma_j\}_{j\in[m]}$ of I, the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for H with lower and upper frame bounds A and B, respectively.

In [17], Li et al. introduced the concept of woven g-frames in Hilbert spaces.

Definition 1.3. [17] A family of g-frames $\{\Lambda_{ij}\}_{i\in I, j\in[m]}$ for a Hilbert space H is said to be woven if there are universal positive constants A and B such that for every partition $\{\sigma_j\}_{j\in[m]}$ of I, the family $\{\Lambda_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a g-frame for H with lower and upper frame bounds A and B, respectively.

In recent years, many mathematicians got significant results by extending the theory of frames from Hilbert spaces to Hilbert C^*-modules. Hilbert C^*-modules are generalizations of Hilbert spaces by allowing the inner product to take values in a C^*-algebra rather than in the field of real or complex numbers. They were introduced and investigated initially by Kaplansky [14]. Frank and Larson [12] introduced the concept of frames in finitely or countably generated Hilbert C^*-modules over a unital C^*-algebra.

In [15], A. Khosravi and B. Khosravi introduced g-frames in Hilbert C^*-modules and observed that they share many useful properties with their corresponding notions in Hilbert spaces. Let $\mathcal{U}$ and $\mathcal{V}$ be finitely or countably generated Hilbert $\mathcal{A}$-modules, and $\{\mathcal{V}_i : i \in I\}$ be a sequence of closed Hilbert submodules of $\mathcal{V}$. Let $End^*_{\mathcal{A}}(\mathcal{U}, \mathcal{V}_i)$ be the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{U}$ to $\mathcal{V}_i$.

Definition 1.4. [15] A sequence $\{\Lambda_i \in End^*_{\mathcal{A}}(\mathcal{U}, \mathcal{V}_i)\colon i \in I\}$ is called a g-frame or a generalized frame in $\mathcal{U}$ with respect to $\{\mathcal{V}_i\colon i \in I\}$ if there exist constants $C, D > 0$ such that for every $f \in \mathcal{U}$,

$$C\langle f, f\rangle \leq \sum_{i\in I}\langle \Lambda_i f, \Lambda_i f\rangle \leq D\langle f, f\rangle. \tag{1.2}$$

Woven frames for finitely or countably generated Hilbert C^*-module were introduced and studied in [13].

Definition 1.5. [13] Let $[m] = \{1, 2, ..., m\}$, $m \in \mathbb{N}$. A family $\Big\{\{\phi_{ij}\}_{i\in I}\Big\}_{j\in[m]}$ of frames for $\mathcal{U}$ is called woven if there exist universal positive constants A and B such that for every partition $\{\sigma_j\}_{j\in[m]}$ of I, the family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is a frame for $\mathcal{U}$ with lower and upper frame bounds A and B, respectively. Each family $\{\phi_{ij}\}_{i\in\sigma_j, j\in[m]}$ is called a weaving.

The above literature motivates us to introduce the notion of woven g-frames in Hilbert C^*-modules. In this paper, we introduce the concept of woven g-frames in Hilbert C^*-modules, and develop their fundamental properties.

2. Woven g-frames

Let I and J be finite or countable index sets and let $\mathbb{N}$ be the set of natural numbers. Throughout this paper, we assume that $\mathcal{U}$ and $\mathcal{V}$ are finitely or countably generated Hilbert $\mathcal{A}$-modules, and $\{\mathcal{V}_i : i \in I\}$ is a sequence of closed Hilbert submodules of $\mathcal{V}$. For each $i \in I$, $End^*_{\mathcal{A}}(\mathcal{U}, \mathcal{V}_i)$ is the collection of all adjointable $\mathcal{A}$-linear maps from $\mathcal{U}$ to $\mathcal{V}_i$ and $End^*_{\mathcal{A}}(\mathcal{U},\mathcal{U})$ is denoted by $End^*_{\mathcal{A}}(\mathcal{U})$.

Now we define woven g-frames in Hilbert C*-modules.

Definition 2.1. Two g-frames $\Lambda = \{\Lambda_i\}_{i\in I}$ and $\Gamma = \{\Gamma_i\}_{i\in I}$ for $\mathcal{U}$ are said to be woven g-frames if there exist universal positive constants A and B such that for any partition σ of I, the family $\{\Lambda_i\}_{i\in\sigma} \bigcup \{\Gamma_i\}_{i\in\sigma^c}$ is a g-frame for $\mathcal{U}$ with lower and upper g-frame bounds A and B, respectively, that is

$$A\langle f, f\rangle \leq \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Gamma_i f, \Gamma_i f\rangle \leq B\langle f, f\rangle, \forall f \in \mathcal{U}. \tag{2.3}$$

Definition 2.2. A family of g-frames $\{\{\Lambda_{ij}\}_{j=1}^{\infty} : i \in I\}$ for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$ is said to be woven g-frames if there exist universal positive constants A and B such that for any partition $\{\sigma_i\}_{i\in I}$ of $\mathbb{N}$, the family $\bigcup_{i\in I}\{\Lambda_{ij}\}_{j\in\sigma_i}$ is a g-frame for $\mathcal{U}$ with lower and upper g-frame bounds A and B, respectively, that is

$$A\langle f, f\rangle \leq \sum_{i\in I}\sum_{j\in\sigma_i}\langle \Lambda_{ij} f, \Lambda_{ij} f\rangle \leq B\langle f, f\rangle, \ \ \forall f \in \mathcal{U}. \tag{2.4}$$

Let $\{\mathcal{V}_i : i \in I\}$ be a sequence of Hilbert $\mathcal{A}$-modules, we define the space

$$\bigoplus_{i\in I}\mathcal{V}_i = \Big\{\{c_{ij}\}_{j\in\sigma_i, i\in I} : c_{ij} \in \mathcal{V}_i \text{ such that } \sum_{j\in\sigma_i, i\in I}\langle c_{ij}, c_{ij}\rangle \text{ is norm convergent in } \mathcal{A}\Big\}.$$

The inner product in $\bigoplus_{i\in I}\mathcal{V}_i$ is defined by

$$\langle\{c_{ij}\}, \{d_{ij}\}\rangle = \sum_{i\in I}\sum_{j\in\sigma_i}\langle c_{ij}, d_{ij}\rangle.$$

Let $\{\{\Lambda_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a family of woven g-frames.
The operator $T\colon \mathcal{U} \to \bigoplus_{i\in I}\mathcal{V}_i$ defined by

$$Tf \ = \ \{\Lambda_{ij} f\}_{j\in\sigma_i, i\in I}$$

is called the analysis operator.
The operator $T^*\colon \bigoplus_{i\in I}\mathcal{V}_i \to \mathcal{U}$ defined by

$$T^*\{c_{ij}\} = \sum_{i\in I}\sum_{j\in\sigma_i}\Lambda^*_{ij} c_{ij}$$

is called the synthesis operator.
By composing T and T^*, we obtain the frame operator $S\colon \mathcal{U} \to \mathcal{U}$ as

$$\begin{aligned} Sf &= T^*Tf \\ &= \sum_{i\in I}\sum_{j\in\sigma_i}\Lambda^*_{ij}\Lambda_{ij} f, \end{aligned}$$

where Λ_{ij}^* is the adjoint operator of Λ_{ij}.

Proposition 2.1. *Let $\{\{\Lambda_{ij}\}_{j=1}^{\infty} : i \in I\}$ be a family of woven g-frames for $\mathcal{U}$. Then the frame operator S is self adjoint, positive, bounded and invertible on $\mathcal{U}$.*

Proof. Since $S^* = (T^*T)^* = T^*T = S$, the frame operator S is self adjoint.
Let $\{\{\Lambda_{ij}\}_{j=1}^{\infty} : i \in I\}$ be woven g-frame for $\mathcal{U}$ with universal lower and upper frame bounds A and B, respectively.
For any $f \in \mathcal{U}$, $Sf = \sum_{i\in I}\sum_{j\in\sigma_i} \Lambda_{ij}^*\Lambda_{ij} f$. Then

$$\begin{aligned} \langle Sf, f\rangle &= \Big\langle \sum_{i\in I}\sum_{j\in\sigma_i} \Lambda_{ij}^*\Lambda_{ij} f, f\Big\rangle \\ &= \sum_{i\in I}\sum_{j\in\sigma_i} \langle \Lambda_{ij} f, \Lambda_{ij} f\rangle. \end{aligned}$$

$\Rightarrow A\langle f, f\rangle \leq \langle Sf, f\rangle \leq B\langle f, f\rangle$.
$\Rightarrow AI \leq S \leq BI$.
Therefore, the frame operator S is positive, bounded and invertible. □

Theorem 2.1. *Let $\{\{\Lambda_{ij}\}_{j=1}^{m} : i \in I\}$ be a sequence of g-Bessel sequences for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$ and with g-Bessel bounds B_j. Then, every weaving is a g-Bessel sequence with bound $\sum_{j=1}^{m} B_j$.*

Proof. Let $[m] = \{1, 2, ..., m\}$, and let $\{\sigma_j\}_{j\in[m]}$ be any partition of I. Then for every $f \in \mathcal{U}$, we have

$$\begin{aligned} \sum_{j=1}^{m}\sum_{i\in\sigma_j} \langle \Lambda_{ij} f, \Lambda_{ij} f\rangle &\leq \sum_{j=1}^{m}\sum_{i\in I} \langle \Lambda_{ij} f, \Lambda_{ij} f\rangle \\ &\leq \sum_{j=1}^{m} B_j\langle f, f\rangle. \end{aligned}$$

Hence the proof. □

Proposition 2.2. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{N}}$ be g-Bessel sequences in $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in \mathbb{N}\}$ with g-Bessel bounds B_1, B_2, respectively. If $J \subset \mathbb{N}$, and $\Lambda_J \equiv \{\Lambda_j\}_{j\in J}$ and $\Gamma_J \equiv \{\Gamma_j\}_{j\in J}$ are woven g-frames, then Λ and Γ are woven g-frames for $\mathcal{U}$.*

Proof. Let A be universal lower g-frame bound for the woven g-frame Λ_J and Γ_J, and let $\sigma \subset \mathbb{N}$ be an arbitrary subset. Then,

$$\begin{aligned} A\langle f, f\rangle &\leq \sum_{j\in\sigma\cap J} \langle \Lambda_j f, \Lambda_j f\rangle + \sum_{j\in\sigma^c\cap J} \langle \Gamma_j f, \Gamma_j f\rangle \\ &\leq \sum_{j\in\sigma} \langle \Lambda_j f, \Lambda_j f\rangle + \sum_{j\in\sigma^c} \langle \Gamma_j f, \Gamma_j f\rangle \\ &\leq (B_1 + B_2)\langle f, f\rangle. \end{aligned}$$

Hence, Λ and Γ are woven g-frames for $\mathcal{U}$. □

Theorem 2.2. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{N}}$ be woven g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$ with universal g-frame bounds A and B. If $J \subset \mathbb{N}$ and*

$$\sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \leq D\langle f, f\rangle$$

for all $f \in \mathcal{U}$ and for some $0 < D < A$. Then $\Lambda_0 \equiv \{\Lambda_i\}_{i\in\mathbb{N}\backslash J}$ and $\Gamma_0 \equiv \{\Gamma_i\}_{i\in\mathbb{N}\backslash J}$ are woven g-frames for $\mathcal{U}$ with universal g-frame bounds $A - D$ and B.

Proof. Let σ be any subset of $\mathbb{N}\backslash J$. We compute

$$\begin{aligned}
&\sum_{j\in\sigma}\langle \Lambda_j f, \Lambda_j f\rangle + \sum_{j\in(\mathbb{N}\backslash J)\backslash\sigma}\langle \Gamma_j f, \Gamma_j f\rangle \\
= \;& \Big(\sum_{j\in\sigma\bigcup J}\langle \Lambda_j f, \Lambda_j f\rangle - \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle\Big) + \sum_{j\in(\mathbb{N}\backslash J)\backslash\sigma}\langle \Gamma_j f, \Gamma_j f\rangle \\
= \;& \Big(\sum_{j\in\sigma\bigcup J}\langle \Lambda_j f, \Lambda_j f\rangle + \sum_{j\in(\mathbb{N}\backslash J)\backslash\sigma}\langle \Gamma_j f, \Gamma_j f\rangle\Big) - \sum_{j\in J}\langle \Lambda_j f, \Lambda_j f\rangle \\
\geq \;& A\langle f, f\rangle - D\langle f, f\rangle = (A - D)\langle f, f\rangle, \forall f \in \mathcal{U}.
\end{aligned}$$

On the other hand, for all $f \in \mathcal{U}$, we have

$$\begin{aligned}
&\sum_{j\in\sigma}\langle \Lambda_j f, \Lambda_j f\rangle + \sum_{j\in(\mathbb{N}\backslash J)\backslash\sigma}\langle \Gamma_j f, \Gamma_j f\rangle \\
\leq \;& \sum_{j\in\sigma\bigcup J}\langle \Lambda_j f, \Lambda_j f\rangle + \sum_{j\in(\mathbb{N}\backslash J)\backslash\sigma}\langle \Gamma_j f, \Gamma_j f\rangle \leq B\langle f, f\rangle.
\end{aligned}$$

Hence, Λ_0 and Γ_0 are woven g-frames for $\mathcal{U}$ with the universal lower and upper g-frame bounds $A - D$ and B, respectively. □

Lemma 2.1. [1] *Let $\mathcal{U}$ and $\mathcal{V}$ be two Hilbert $\mathcal{A}$-modules over a C^*-algebra $\mathcal{A}$, and $T \in End^*_{\mathcal{A}}(\mathcal{U}, \mathcal{V})$. Then the following statements are equivalent:*

1. *T is surjective.*
2. *T^* is bounded below with respect to norm i.e there exists $m > 0$ such that $\|T^* f\| \geq m\|f\|$ for all $f \in \mathcal{U}$.*
3. *T^* is bounded below with respect to inner product i.e there exists $m > 0$ such that $\langle T^* f, T^* f\rangle \geq m\langle f, f\rangle$ for all $f \in \mathcal{U}$.*

Lemma 2.2. [18] *Let $\mathcal{U}$ and $\mathcal{V}$ be Hilbert $\mathcal{A}$-modules over a C^*-algebra $\mathcal{A}$, and let $T : \mathcal{U} \to \mathcal{V}$ be a linear map. Then the following conditions are equivalent:*

1. *The operator T is bounded and $\mathcal{A}$-linear.*
2. *There exists $k \geq 0$ such that $\langle Tx, Tx\rangle \leq k\langle x, x\rangle$ for all $x \in \mathcal{U}$.*

Theorem 2.3. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{N}}$ be a pair of g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in \mathbb{N}\}$. Then for every partition σ of $\mathbb{N}$, Λ and Γ are woven g-frames for $\mathcal{U}$ with universal lower and upper g-frame bounds A and B, respectively, if and only if*

$$A\|\langle f, f\rangle\| \leq \|\sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Gamma_i f, \Gamma_i f\rangle\| \leq B\|\langle f, f\rangle\|$$

for all $f \in \mathcal{U}$.

Proof. ($\Longrightarrow$) Obvious.
Now assume that there exist constants $0 < A, B < \infty$ such that for all $f \in \mathcal{U}$

$$A\|\langle f, f\rangle\| \leq \|\sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Gamma_i f, \Gamma_i f\rangle\| \leq B\|\langle f, f\rangle\|. \tag{2.5}$$

We prove that Λ and Γ are g-woven frames for $\mathcal{U}$ with the universal lower and upper g-frame bounds A and B, respectively.
As S is positive, self adjoint and invertible operator. We have

$$\langle S^{\frac{1}{2}} f, S^{\frac{1}{2}} f\rangle = \langle Sf, f\rangle = \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Gamma_i f, \Gamma_i f\rangle.$$

From equation (2.5), we have

$$\sqrt{A}\|f\| \leq \|S^{\frac{1}{2}} f\| \leq \sqrt{B}\|f\|.$$

By using Lemma 2.1, we have

$$\langle S^{\frac{1}{2}} f, S^{\frac{1}{2}} f\rangle = \langle Sf, f\rangle \geq A\langle f, f\rangle.$$

Since $S^{\frac{1}{2}}$ is bounded and $\mathcal{A}$-linear, by using Lemma 2.2, we have

$$\langle S^{\frac{1}{2}} f, S^{\frac{1}{2}} f\rangle = \langle Sf, f\rangle \leq B\langle f, f\rangle.$$

□

THEOREM 2.4. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{N}}$ be a pair of g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in \mathbb{N}\}$ with g-frame bounds A_1, B_1 and A_2, B_2, respectively. Assume that there are constants $0 < \lambda_1, \lambda_2, \mu < 1$ such that*

$$\lambda_1\sqrt{B_1} + \lambda_2\sqrt{B_2} + \mu \leq \frac{A_1}{2(\sqrt{B_1} + \sqrt{B_2})}$$

and

$$\begin{aligned} \|\sum_{i\in\mathbb{N}}\langle (\Lambda_i^* - \Gamma_i^*)f_i, (\Lambda_i^* - \Gamma_i^*)f_i\rangle\|^{\frac{1}{2}} &\leq \lambda_1\|\sum_{i\in\mathbb{N}}\langle \Lambda_i^* f_i, \Lambda_i^* f_i\rangle\|^{\frac{1}{2}} \\ &+ \lambda_2\|\sum_{i\in\mathbb{N}}\langle \Gamma_i^* f_i, \Gamma_i^* f_i\rangle\|^{\frac{1}{2}} + \mu\|\langle\{f_i\}, \{f_i\}\rangle\|^{\frac{1}{2}} \end{aligned} \tag{2.6}$$

for all $\{f_i\}_{i\in\mathbb{N}} \in \big(\bigoplus_{i\in\mathbb{N}} V_i\big)$. Then, Λ and Γ are woven g-frames with universal lower and upper frame bounds $\frac{A_1}{2}$ and $B_1 + B_2$, respectively.

Proof. Let T and R be the synthesis operator for the g-frames $\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\{\Gamma_i\}_{i\in\mathbb{N}}$, respectively. $T\colon \bigoplus_{i\in\mathbb{N}} \mathcal{V}_i \to \mathcal{U}$ is defined as

$$T\{f_i\} = \sum_{i\in\mathbb{N}} \Lambda_i^* f_i,$$

and $R\colon \bigoplus_{i\in\mathbb{N}} \mathcal{V}_i \to \mathcal{U}$ is defined as

$$R\{f_i\} = \sum_{i\in\mathbb{N}} \Gamma_i^* f_i.$$

For each $\sigma \subset \mathbb{N}$, define the bounded operators

$$T_\sigma, R_\sigma \colon (\bigoplus_{i\in\sigma} V_i) \to \mathcal{U}$$

as $T_\sigma(\{f_i\}) = \sum_{i\in\sigma} \Lambda_i^* f_i$ and $R_\sigma(\{f_i\}) = \sum_{i\in\sigma} \Gamma_i^* f_i$.
We note that $\|T_\sigma\| \leq \|T\|$, $\|R_\sigma\| \leq \|R\|$ and $\|T_\sigma - R_\sigma\| \leq \|T - R\|$.
As we know $\|f\|^2 = \|\langle f, f\rangle\|, \forall f \in \mathcal{U}$ and using equation (2.6), we have

$$\begin{aligned} & \lambda_1\|T(\{f_i\}_{i\in\mathbb{N}})\| + \lambda_2\|R(\{f_i\}_{i\in\mathbb{N}})\| + \mu\|\{f_i\}_{i\in\mathbb{N}}\| \\ \geq\ & \|\sum_{i\in\mathbb{N}}\langle(\Lambda_i^* - \Gamma_i^*)f_i, (\Lambda_i^* - \Gamma_i^*)f_i\rangle\|^{\frac{1}{2}} = \|(T-R)(\{f_i\}_{i\in\mathbb{N}})\|. \end{aligned}$$

This gives $\|T - R\| \leq \lambda_1\|T\| + \lambda_2\|R\| + \mu$.
Using this, for any $\sigma \subset \mathbb{N}$, we compute

$$\begin{aligned} \|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f - \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| &= \|T_\sigma(\{\Lambda_i f\}_{i\in\sigma}) - R_\sigma(\{\Gamma_i f\}_{i\in\sigma})\| \\ &= \|T_\sigma T_\sigma^* f - R_\sigma R_\sigma^* f\| \\ &= \|T_\sigma T_\sigma^* f - T_\sigma R_\sigma^* f + T_\sigma R_\sigma^* f - R_\sigma R_\sigma^* f\| \\ &\leq \|(T_\sigma T_\sigma^* - T_\sigma R_\sigma^*)f\| + \|(T_\sigma R_\sigma^* - R_\sigma R_\sigma^*)f\| \\ &\leq \|T_\sigma\|\|T_\sigma^* - R_\sigma^*\|\|f\| + \|T_\sigma - R_\sigma\|\|R_\sigma^*\|\|f\| \\ &\leq \|T\|\|T - R\|\|f\| + \|T - R\|\|R\|\|f\| \\ &\leq (\lambda_1\|T\| + \lambda_2\|R\| + \mu)(\|T\| + \|R\|)\|f\| \\ &\leq (\lambda_1\|T\| + \lambda_2\|R\| + \mu)(\sqrt{B_1} + \sqrt{B_2})\|f\| \\ &< \frac{A_1}{2(\sqrt{B_1} + \sqrt{B_2})}(\sqrt{B_1} + \sqrt{B_2})\|f\| \\ &= \frac{A_1}{2}\|f\|. \end{aligned} \tag{2.7}$$

Now, from the equation (2.7), it follows that

$$\begin{aligned} & \|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_i f + \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| \\ =\ & \|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_i f + \sum_{i\in\sigma}\Lambda_i^*\Lambda_i f - \sum_{i\in\sigma}\Lambda_i^*\Lambda_i f + \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| \\ =\ & \|\sum_{i\in\mathbb{N}}\Lambda_i^*\Lambda_i f + \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f - \sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\| \\ \geq\ & \|\sum_{i\in\mathbb{N}}\Lambda_i^*\Lambda_i f\| - \|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f - \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| \\ \geq\ & A_1\|f\| - \|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f - \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| \\ \geq\ & A_1\|f\| - \frac{A_1}{2}\|f\| \\ =\ & \frac{A_1}{2}\|f\|. \end{aligned}$$

This gives universal lower g-frame bound. By using Theorem 2.1, we get $B_1 + B_2$ as universal upper g-frame bound. Hence, Λ and Γ are woven g-frames. $\square$

THEOREM 2.5. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ and $\Gamma = \{\Gamma_i\}_{i\in\mathbb{N}}$ be a pair of g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in \mathbb{N}\}$ with g-frame bounds A_1, B_1 and A_2, B_2, respectively. Assume that there are constants $0 < \lambda, \mu, \gamma < 1$ such that*

$$\lambda B_1 + \mu B_2 + \gamma < A_1$$

and

$$\begin{aligned} & \|\sum_{i\in\sigma}\langle(\Lambda_i^*\Lambda_i - \Gamma_i^*\Gamma_i)f, (\Lambda_i^*\Lambda_i - \Gamma_i^*\Gamma_i)f\rangle\|^{\frac{1}{2}} \\ \leq\ & \lambda\|\sum_{i\in\sigma}\langle\Lambda_i^*\Lambda_i f, \Lambda_i^*\Lambda_i f\rangle\|^{\frac{1}{2}} + \mu\|\sum_{i\in\sigma}\langle\Gamma_i^*\Gamma_i f, \Gamma_i^*\Gamma_i f\rangle\|^{\frac{1}{2}} \\ +\ & \gamma(\sum_{i\in\sigma}\|\Lambda_i f\|^2)^{\frac{1}{2}}. \end{aligned} \tag{2.8}$$

for all $f \in \mathcal{U}$ and for every $\sigma \subset \mathbb{N}$. Then, Λ and Γ are woven g-frames with universal g-frame bounds $(A_1 - \lambda B_1 - \mu B_2 - \gamma\sqrt{B_1})$ and $(B_1 + \lambda B_1 + \mu B_2 + \gamma\sqrt{B_1})$.

Proof. For any $\sigma \subset \mathbb{N}$, we use the fact that for $f \in \mathcal{U}$,
$\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\| \leq B_1\|f\|$ and $\|\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| \leq B_2\|f\|$
and as we know that $\|f\|^2 = \|\langle f, f\rangle\|, \forall f \in \mathcal{U}$, (2.8) implies

$$\begin{aligned} & \|\sum_{i\in\sigma}(\Lambda_i^*\Lambda_i - \Gamma_i^*\Gamma_i)f\| \\ \leq\ & \lambda\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\| + \mu\|\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| + \gamma(\sum_{i\in\sigma}\|\Lambda_i f\|^2)^{\frac{1}{2}}. \end{aligned} \tag{2.9}$$

We compute

$$\begin{aligned} & \|\sum_{i\in\sigma^c}\Lambda_i^*\Lambda_i f + \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| \\ =\ & \|\sum_{i\in\mathbb{N}}\Lambda_i^*\Lambda_i f + \sum_{i\in\sigma}\Gamma_i^*\Gamma_i f - \sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\| \\ \geq\ & \|\sum_{i\in\mathbb{N}}\Lambda_i^*\Lambda_i f\| - \|\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f - \sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\| \end{aligned}$$

$$\begin{aligned} \geq\ & A_1\|f\| - \|\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f - \sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\| \\ \geq\ & A_1\|f\| - \lambda\|\sum_{i\in\sigma}\Lambda_i^*\Lambda_i f\| - \mu\|\sum_{i\in\sigma}\Gamma_i^*\Gamma_i f\| - \gamma(\sum_{i\in\sigma}\|\Lambda_i f\|^2)^{\frac{1}{2}} \\ \geq\ & (A_1 - \lambda B_1 - \mu B_2 - \gamma\sqrt{B_1})\|f\|, \end{aligned}$$

and

$$\begin{aligned}
& \| \sum_{i\in\sigma^c} \Lambda_i^*\Lambda_i f + \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f \| \\
= \ & \| \sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i f + \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f \| \\
\leq \ & \| \sum_{i\in\mathbb{N}} \Lambda_i^*\Lambda_i f \| + \| \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f - \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f \| \\
\leq \ & B_1\|f\| + \lambda \| \sum_{i\in\sigma} \Lambda_i^*\Lambda_i f \| + \mu \| \sum_{i\in\sigma} \Gamma_i^*\Gamma_i f \| + \gamma (\sum_{i\in\sigma} \|\Lambda_i f\|^2)^{\frac{1}{2}} \\
\leq \ & (B_1 + \lambda B_1 + \mu B_2 + \gamma\sqrt{B_1})\|f\|.
\end{aligned}$$

Therefore, Λ and Γ are woven g-frames with the universal lower and upper bounds $(A_1 - \lambda B_1 - \mu B_2 - \gamma\sqrt{B_1})$ and $(B_1 + \lambda B_1 + \mu B_2 + \gamma\sqrt{B_1})$, respectively. □

THEOREM 2.6. *For $i \in I$, let $\Lambda_i = \{\Lambda_{ij}\}_{j\in J}$ be a family of g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$ with bounds A_i and B_i. For any $\sigma \subset J$ and a fix $t \in I$, let $P_i^\sigma(f) = \sum_{j\in\sigma} \Lambda_{ij}^*\Lambda_{ij} f - \sum_{j\in\sigma} \Lambda_{tj}^*\Lambda_{tj} f$ for $i \neq t$. If P_i^σ is a positive linear operator, then the family of g-frames $\{\Lambda_i\}_{i\in I}$ is g-woven.*

Proof. Let $\{\sigma_i\}_{i\in[m]}$ be any partition of J. Then, for every $f \in \mathcal{U}$, a fix $t \in I$ and $j \in \sigma_i$, we have

$$\begin{aligned}
& \sum_{j\in\sigma_i} \langle \Lambda_{tj}^*\Lambda_{tj} f, f\rangle \\
= \ & \sum_{j\in\sigma_i} \big\langle \Lambda_{ij}^*\Lambda_{ij} f - P_i^\sigma(f), f \big\rangle \\
\leq \ & \sum_{j\in\sigma_i} \big\langle \Lambda_{ij}^*\Lambda_{ij} f, f \big\rangle. \quad \text{(As } P_i^\sigma \text{ is a positive linear operator)}
\end{aligned} \tag{2.10}$$

Now, using (2.10) we have

$$\begin{aligned}
A_t\langle f, f\rangle \ \leq \ & \sum_{j\in J} \langle \Lambda_{tj}^*\Lambda_{tj} f, f\rangle \\
= \ & \sum_{j\in\sigma_1} \langle \Lambda_{tj}^*\Lambda_{tj} f, f\rangle + ... + \sum_{j\in\sigma_i} \langle \Lambda_{tj}^*\Lambda_{tj} f, f\rangle + ... + \sum_{j\in\sigma_m} \langle \Lambda_{tj}^*\Lambda_{tj} f, f\rangle \\
\leq \ & \sum_{j\in\sigma_1} \langle \Lambda_{1j}^*\Lambda_{1j} f, f\rangle + ... + \sum_{j\in\sigma_i} \langle \Lambda_{ij}^*\Lambda_{ij} f, f\rangle + ... + \sum_{j\in\sigma_m} \langle \Lambda_{mj}^*\Lambda_{mj} f, f\rangle \\
\leq \ & (B_1 + ... + B_i + ... + B_m)\langle f, f\rangle \\
= \ & \sum_{i\in I} B_i \langle f, f\rangle.
\end{aligned}$$

This implies that

$$A_t\langle f, f\rangle \leq \sum_{i\in I} \sum_{j\in\sigma_i} \langle \Lambda_{ij}^*\Lambda_{ij} f, f\rangle \leq \sum_{i\in I} B_i \langle f, f\rangle.$$

□

THEOREM 2.7. *For each $j \in [m]$, let $\Lambda_i = \{\Lambda_{ij}\}_{i\in I}$ be a family of g-frames for $\mathcal{U}$ with bounds A_j and B_j. Suppose there exists $K > 0$ such that*

$$\begin{aligned} &\sum_{i\in J} \|\langle(\Lambda_{ij}-\Lambda_{il})f,(\Lambda_{ij}-\Lambda_{il})f\rangle\| \\ \leq\ & K\min\big\{\sum_{i\in J}\|\langle\Lambda_{ij}f,\Lambda_{ij}f\rangle\|,\sum_{i\in J}\|\langle\Lambda_{il}f,\Lambda_{il}f\rangle\|\big\} \quad (j,l\in[m], j\neq l)\end{aligned}$$

for all $f \in \mathcal{U}$ and for all subsets $J \subset I$. Then the family of g-frames $\{\{\Lambda_{ij}\}_{i\in I} : j \in [m]\}$ is woven with universal frame bounds
$\dfrac{\sum_{j\in[m]} A_j}{2(m-1)(K+1)+1}$ *and* $\displaystyle\sum_{j\in[m]} B_j$.

Proof. Let $\{\sigma_j\}_{j\in[m]}$ be any partition of I. For the lower frame inequality, we have

$$\begin{aligned} &\sum_{j\in[m]} A_j\|\langle f,f\rangle\| \\ =\ & A_1\|\langle f,f\rangle\| + ... + A_m\|\langle f,f\rangle\| \\ \leq\ & \sum_{i\in I}\|\langle\Lambda_{i1}f,\Lambda_{i1}f\rangle\| + ... + \sum_{i\in I}\|\langle\Lambda_{im}f,\Lambda_{im}f\rangle\| \\ =\ & \big(\sum_{i\in\sigma_1}\|\langle\Lambda_{i1}f,\Lambda_{i1}f\rangle\| + ... + \sum_{i\in\sigma_m}\|\langle\Lambda_{i1}f,\Lambda_{i1}f\rangle\|\big) + ... \\ +\ & \big(\sum_{i\in\sigma_1}\|\langle\Lambda_{im}f,\Lambda_{im}f\rangle\| + ... + \sum_{i\in\sigma_m}\|\langle\Lambda_{im}f,\Lambda_{im}f\rangle\|\big) \\ \leq\ & \big[\sum_{i\in\sigma_1}\|\langle\Lambda_{i1}f,\Lambda_{i1}f\rangle\| + 2\big(\sum_{i\in\sigma_2}\|\langle(\Lambda_{i1}-\Lambda_{i2})f,(\Lambda_{i1}-\Lambda_{i2})f\rangle\| + \sum_{i\in\sigma_2}\|\langle\Lambda_{i2}f,\Lambda_{i2}f\rangle\|\big) \\ +\ & ... + 2\big(\sum_{i\in\sigma_m}\|\langle(\Lambda_{i1}-\Lambda_{im})f,(\Lambda_{i1}-\Lambda_{im})f\rangle\| + \sum_{i\in\sigma_m}\|\langle\Lambda_{im}f,\Lambda_{im}f\rangle\|\big)\big] + ... \\ +\ & \big[2\big(\sum_{i\in\sigma_1}\|\langle(\Lambda_{im}-\Lambda_{i1})f,(\Lambda_{im}-\Lambda_{i1})f\rangle\| + \sum_{i\in\sigma_1}\|\langle\Lambda_{i1}f,\Lambda_{i1}f\rangle\|\big) + ... \\ +\ & 2\big(\sum_{i\in\sigma_{m-1}}\|\langle(\Lambda_{im}-\Lambda_{i(m-1)})f,(\Lambda_{im}-\Lambda_{i(m-1)})f\rangle\| + \sum_{i\in\sigma_m}\|\langle\Lambda_{i(m-1)}f,\Lambda_{i(m-1)}f\rangle\|\big) \\ +\ & \sum_{i\in\sigma_m}\|\langle\Lambda_{im}f,\Lambda_{im}f\rangle\|\big]\end{aligned}$$

$$\begin{aligned}
&\leq && \big[\sum_{i\in\sigma_1}\|\langle\Lambda_{i1}f,\Lambda_{i1}f\rangle\|+2\big(K\sum_{i\in\sigma_2}\|\langle\Lambda_{i2}f,\Lambda_{i2}f\rangle\|+\sum_{i\in\sigma_2}\|\langle\Lambda_{i2}f,\Lambda_{i2}f\rangle\|\big)+...\\
&+ && 2\big(K\sum_{i\in\sigma_m}\|\langle\Lambda_{im}f,\Lambda_{im}f\rangle\|+\sum_{i\in\sigma_m}\|\langle\Lambda_{im}f,\Lambda_{im}f\rangle\|\big)\big]+...\\
&+ && \big[2\big(K\sum_{i\in\sigma_1}\|\langle\Lambda_{i1}f,\Lambda_{i1}f\rangle\|+\sum_{i\in\sigma_1}\|\langle\Lambda_{i1}f,\Lambda_{i1}f\rangle\|\big)+...\\
&+ && 2\big(K\sum_{i\in\sigma_{m-1}}\|\langle\Lambda_{i(m-1)}f,\Lambda_{i(m-1)})f\rangle\|+\sum_{i\in\sigma_{(m-1)}}\|\langle\Lambda_{i(m-1)}f,\Lambda_{i(m-1)}f\rangle\|\big)\\
&+ && \sum_{i\in\sigma_m}\|\langle\Lambda_{im}f,\Lambda_{im}f\rangle\|\big]\\
&= && \sum_{i\in\sigma_1}\|\langle\Lambda_{i1}f,\Lambda_{i1}f\rangle\|+...+\sum_{i\in\sigma_m}\|\langle\Lambda_{im}f,\Lambda_{im}f\rangle\|\\
&+ && (m-1)2(K+1)\Big(\sum_{i\in\sigma_1}\|\langle\Lambda_{i1}f,\Lambda_{i1}f\rangle\|+...+\sum_{i\in\sigma_m}\|\langle\Lambda_{im}f,\Lambda_{im}f\rangle\|\Big)\\
&= && \Big[2(m-1)(K+1)+1\Big]\sum_{j\in[m]}\sum_{i\in\sigma_j}\|\langle\Lambda_{ij}f,\Lambda_{ij}f\rangle\|
\end{aligned}$$

for all $f\in\mathcal{U}$. From Theorem 2.1, we know that $\{\{\Lambda_{ij}\}_{i\in I}:j\in[m]\}$ satisfies upper frame inequality with universal upper frame bound $\sum_{j\in[m]}B_j$. Hence, for all $f\in\mathcal{U}$, we have

$$\frac{\sum_{j\in[m]}A_j}{2(m-1)(K+1)+1}\|\langle f,f\rangle\|\leq\sum_{j\in[m]}\sum_{i\in\sigma_j}\|\langle\Lambda_{ij}f,\Lambda_{ij}f\rangle\|\leq\sum_{j\in[m]}B_j\|\langle f,f\rangle\|.$$

□

PROPOSITION 2.3. *Let $\{\Lambda_{ij}\}_{i\in I,j\in[m]}$ be a family of woven g-Bessel sequences for $\mathcal{U}$ with respect to $\{\mathcal{V}_i:i\in I\}$ and with g-Bessel with bound B. Then, $\{\Lambda_{ij}T\}_{i\in I,j\in[m]}$ is also woven g-Bessel sequence with bound $B\|T\|^2$ for every $T\in L(\mathcal{U})$.*

Proof. Suppose that $\{\Lambda_{ij}\}_{i\in I,j\in[m]}$ is a family of woven g-Bessel sequence for $\mathcal{U}$ with respect to $\{\mathcal{V}_i:i\in I\}$ and with g-Bessel bound B. Then for any partition $\{\sigma_j\}_{j\in[m]}$ of I, we have

$$\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle\Lambda_{ij}f,\Lambda_{ij}f\rangle\quad\leq\quad B\langle f,f\rangle.$$

Now

$$\begin{aligned}
\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle\Lambda_{ij}Tf,\Lambda_{ij}Tf\rangle &\leq B\langle Tf,Tf\rangle\\
&\leq B\|T\|^2\langle f,f\rangle,\ \forall f\in\mathcal{U}.
\end{aligned}$$

Hence the proof. □

THEOREM 2.8. *Let $\{\Lambda_{ij}\}_{i\in I, j\in[m]}$ be a family of g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in I\}$. Then $\{\Lambda_{ij}\}_{i\in I, j\in[m]}$ is a woven g-Bessel sequence with bound D if and only if*

$$\|\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle\Lambda_{ij}f, \Lambda_{ij}f\rangle\| \leq D\|f\|^2, \forall f \in \mathcal{U}$$

holds for any partition $\{\sigma_j\}_{j\in[m]}$ of I.

Proof. ($\Longrightarrow$) Obvious.
On the other hand, we define a linear operator $T\colon \mathcal{U} \to \bigoplus_{i\in I}\mathcal{V}_i$ defined as

$$Tf \;=\; \sum_{j=1}^{m}\sum_{i\in\sigma_j}\Lambda_{ij}fe_{ij}$$

for any partition $\{\sigma_j\}_{j\in[m]}$ of I, where $\{e_{ij}\}_{i\in\sigma_j, j\in[m]}$ are the standard orthonormal bases for $\mathcal{V}_i$.
Then

$$\|Tf\|^2 = \|\langle Tf, Tf\rangle\| = \|\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle\Lambda_{ij}f, \Lambda_{ij}f\rangle\| \leq D\|f\|^2.$$

This implies that $\|Tf\| \leq \sqrt{D}\|f\|$. Hence T is bounded. It is obvious that T is $\mathcal{A}$-linear. Then by Lemma 2.2, we have

$$\langle Tf, Tf\rangle \leq D\langle f, f\rangle.$$

Equivalently, $\sum_{j=1}^{m}\sum_{i\in\sigma_j}\langle\Lambda_{ij}f, \Lambda_{ij}f\rangle \leq D\langle f, f\rangle$, as desired. □

EXAMPLE 2.1. Let $\mathcal{A} = l^\infty$, $\mathcal{U} = C_0$ be the Hilbert $\mathcal{A}$-module of the set of all null sequences equipped with the $\mathcal{A}$-inner product

$$\langle u, v\rangle = uv^* = \{u_iv_i^*\}_{i=1}^{\infty} = \{u_i\overline{v_i}\}_{i=1}^{\infty}$$

for any $u = \{u_i\}_{i=1}^{\infty} \in \mathcal{U}$ and $v = \{v_i\}_{i=1}^{\infty} \in \mathcal{U}$.
Let $j \in J = \mathbb{N}$ and define $A_j \in B(\mathcal{U})$ by $A_j\big(\{f_i\}_{i\in\mathbb{N}}\big) = \{\delta_{ij}f_j\}_{i\in\mathbb{N}}$ $\forall\{f_i\}_{i\in\mathbb{N}} \in \mathcal{U}$.
Let $\Lambda = \{\Lambda_j\}_{j=1}^{\infty}$ and $\Gamma = \{\Gamma_j\}_{j=1}^{\infty}$ be defined as follows:

$$\begin{aligned}\{\Lambda_j\}_{j=1}^{\infty} &= \{A_1 + A_2, A_1 + A_2, 0, 0, 0, ...\}\\ \{\Gamma_j\}_{j=1}^{\infty} &= \{0, 0, A_3, A_4, A_5, ...\}.\end{aligned}$$

Let $f = \{f_1, f_2, f_3, ...\} \in \mathcal{U}$. Then $\langle f, f\rangle = \{f_1f_1^*, f_2f_2^*, f_3f_3^*, ...\}$. Here partial ordering $'\leq'$ means pointwise comparision.
For any subset σ of $\mathbb{N}$, we have

$$\sum_{j\in\sigma}\langle\Lambda_jf, \Lambda_jf\rangle + \sum_{j\in\sigma^c}\langle\Gamma_jf, \Gamma_jf\rangle \leq 2\langle f, f\rangle.$$

On the other hand, it is clear that

$$\langle f, f\rangle \;\leq\; \sum_{j\in\sigma}\langle\Lambda_jf, \Lambda_jf\rangle + \sum_{j\in\sigma^c}\langle\Gamma_jf, \Gamma_jf\rangle.$$

Hence Λ and Γ are woven g-frames with universal lower and upper frame bounds 1 and 2, respectively.

THEOREM 2.9. *Let $\Lambda = \{\Lambda_i\}_{i\in\mathbb{N}}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_i : i \in \mathbb{N}\}$ with upper and lower g-frame bounds A and B, respectively. Suppose S is the g-frame operator of $\{\Lambda_i\}_{i\in\mathbb{N}}$ such that $S^{-1}\Lambda_i$ is self adjoint for all $i \in \mathbb{N}$. Then $\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\{\Lambda_i^* S^{-1}\}_{i\in\mathbb{N}}$ are woven g-frames for $\mathcal{U}$.*

Proof. Let σ be any partition of $\mathbb{N}$. Since S^{-1} and $S^{-1}\Lambda_i$ are self adjoint, we have

$$\begin{aligned}
A\langle f, f\rangle &\leq \sum_{i\in\mathbb{N}}\langle \Lambda_i f, \Lambda_i f\rangle \\
&= \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Lambda_i f, \Lambda_i f\rangle \\
&= \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle SS^{-1}\Lambda_i f, SS^{-1}\Lambda_i f\rangle \\
&\leq \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\|S\|^2\langle S^{-1}\Lambda_i f, S^{-1}\Lambda_i f\rangle \\
&\leq \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + B^2\sum_{i\in\sigma^c}\langle (S^{-1}\Lambda_i)^* f, (S^{-1}\Lambda_i)^* f\rangle \\
&= \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + B^2\sum_{i\in\sigma^c}\langle \Lambda_i^* S^{-1} f, \Lambda_i^* S^{-1} f\rangle \\
&\leq \max\{1, B^2\}\big(\sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Lambda_i^* S^{-1} f, \Lambda_i^* S^{-1} f\rangle\big).
\end{aligned}$$

This implies that

$$\frac{A}{\max\{1, B^2\}}\langle f, f\rangle \leq \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Lambda_i^* S^{-1} f, \Lambda_i^* S^{-1} f\rangle.$$

Thus $\dfrac{A}{\max\{1, B^2\}}$ is a universal lower g-frame bound. To find a universal upper g-frame bound, we compute

$$\begin{aligned}
&\sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle \Lambda_i^* S^{-1} f, \Lambda_i^* S^{-1} f\rangle \\
&= \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle (S^{-1}\Lambda_i)^* f, (S^{-1}\Lambda_i)^* f\rangle
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\langle S^{-1}\Lambda_i f, S^{-1}\Lambda_i f\rangle \\
&\leq \sum_{i\in\sigma}\langle \Lambda_i f, \Lambda_i f\rangle + \sum_{i\in\sigma^c}\|S^{-1}\|^2\langle \Lambda_i f, \Lambda_i f\rangle
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i\in\sigma}\langle\Lambda_i f,\Lambda_i f\rangle+\frac{1}{A^2}\sum_{i\in\sigma^c}\langle\Lambda_i f,\Lambda_i f\rangle\\
&\leq \max\{1,\frac{1}{A^2}\}\sum_{i\in\mathbb{N}}\langle\Lambda_i f,\Lambda_i f\rangle\\
&\leq B\max\{1,\frac{1}{A^2}\}\langle f,f\rangle.
\end{aligned}
$$

Hence, $\{\Lambda_i\}_{i\in\mathbb{N}}$ and $\{\Lambda_i^* S^{-1}\}_{i\in\mathbb{N}}$ are woven g-frames for $\mathcal{U}$ with universal lower g-frame bound $\dfrac{A}{\max\{1,B^2\}}$ and universal upper g-frame bound $B\max\{1,\dfrac{1}{A^2}\}$. □

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