Korean J. Math. **29** (2021), No. 1, pp. 91–103
http://dx.doi.org/10.11568/kjm.2021.29.1.91

UNIFIED INTEGRAL OPERATOR INEQUALITIES VIA CONVEX COMPOSITION OF TWO FUNCTIONS

Lakshmi Narayan Mishra, Ghulam Farid*, and Kahkashan Mahreen

Abstract. In this paper we have established inequalities for a unified integral operator by using convexity of composition of two functions. The obtained results are directly connected to bounds of various fractional and conformable integral operators which are already known in literature. A generalized Hadamard integral inequality is obtained which further leads to its various versions for associated fractional integrals. Further, some implicated results are discussed.

1. Introduction and Preliminary Results

The subject of fractional calculus has a lot of applications in science and engineering. Like in biological population models, signal processing, optics and electromagnetic, use of fractional calculus is very helpful (see [3, 5] and references therein). Specially, fractional integral inequalities have been studied by many authors using convex functions as a tool.
The goal of this paper is to study a unified integral operator for convex composition of two functions. The considered operator has direct connection with several fractional and conformable integral operators. We start with the definitions of generalized Riemann-Liouville fractional integral operators.

Definition 1. [13] Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order μ where $\mathbb{R}(\mu) > 0$ are defined by:

$$ {}_g^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (g(x)-g(t))^{\mu-1}g'(t)f(t)dt,\ x > a \tag{1.1}$$

and

$$ {}_g^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (g(t)-g(x))^{\mu-1}g'(t)f(t)dt,\ \ x < b, \tag{1.2}$$

where $\Gamma(.)$ is the gamma function.

Received December 3, 2020. Accepted February 16, 2021. Published online March 30, 2021.
2010 Mathematics Subject Classification: 26D10, 31A10, 26A33.
Key words and phrases: Convex function, Bounds, Integral operator, Fractional integrals.
*Corresponding author.

Definition 2. [14] Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order $\mu, k > 0$ are defined by:

$$(1.3)\qquad {}_g^{\mu}I_{a^+}^{k} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_a^x (g(x) - g(t))^{\frac{\mu}{k}-1} g'(t) f(t) dt,\ x > a$$

and

$$(1.4)\qquad {}_g^{\mu}I_{b^-}^{k} f(x) = \frac{1}{k\Gamma_k(\mu)} \int_x^b (g(t) - g(x))^{\frac{\mu}{k}-1} g'(t) f(t) dt,\ \ x < b,$$

where $\Gamma_k(.)$ [16] is defined as follows:

$$(1.5)\qquad \Gamma_k(x) = \int_0^{\infty} t^{x-1} e^{-\frac{t^k}{k}} dt,\ \Re(x) > 0.$$

A generalized fractional integral operator containing an extended Mittag-Leffler function is defined as follows:

Definition 3. [1] Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ with $p \geq 0$, $\delta > 0$ and $0 < k \leq \delta + \Re(\mu)$. Let $f \in L_1[a, b]$ and $x \in [a, b]$. Then the generalized fractional integral operators $\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f$ and $\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f$ are defined by:

$$(1.6)\qquad \left(\epsilon_{\mu,\alpha,l,\omega,a^+}^{\gamma,\delta,k,c} f\right)(x;p) = \int_a^x (x-t)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(x-t)^{\mu};p) f(t) dt,$$

and

$$(1.7)\qquad \left(\epsilon_{\mu,\alpha,l,\omega,b^-}^{\gamma,\delta,k,c} f\right)(x;p) = \int_x^b (t-x)^{\alpha-1} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(t-x)^{\mu};p) f(t) dt,$$

where

$$(1.8)\qquad E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}}$$

is the extended generalized Mittag-Leffler function.

Recently, Farid defined a unified integral operator as follows:

Definition 4. [15] Let $f, g : [a, b] \longrightarrow \mathbb{R}$, $0 < a < b$, be the functions such that f be positive and $f \in L_1[a, b]$, and g be differentiable and strictly increasing. Also let $\frac{\phi}{x}$ be an increasing function on $[a, \infty)$ and $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then for $x \in [a, b]$ the left and right integral operators are defined by

$$(1.9)\qquad ({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f)(x, \omega; p) = \int_a^x K_x^y(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) f(y) d(g(y))$$

and

$$(1.10)\qquad ({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f)(x, \omega; p) = \int_x^b K_y^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) f(y) d(g(y)),$$

where the involved kernel is defined by

$$(1.11)\qquad K_x^y(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) = \frac{\phi(g(x) - g(y))}{g(x) - g(y)} E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x) - g(y))^{\mu}; p).$$

For suitable settings of functions ϕ, g and certain values of parameters included in Mittag-Leffler function (1.8), some interesting consequences can be obtained which are comprised in [15, Remarks 6 & 7]. Many authors have utilized fractional and conformable integrals to obtain interesting generalized results, we refer readers to [9–11, 17, 19, 21, 22, 25, 27, 28]. To derive the results for integral operators (1.9) and (1.10), we need to recall the following definitions:

Definition 5. [23] A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$, where I is an interval in $\mathbb{R}$ is called convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{1.12}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Lemma 1. *[23] Let $f : I \to \mathbb{R}$ be convex and increasing function and $g : J \to \mathbb{R}$, with $Rang(g) \subseteq I$, be convex. Then the composite function $f \circ g$ is convex on J.*

Lemma 2. *[6] Let $f : [a,b] \to \mathbb{R}$ be a convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq f(x), \quad x \in [a,b]. \tag{1.13}$$

Next, we give a property of the kernel given in (1.11), which will be useful for finding the results of this paper.

P: Let g and $\frac{\phi}{I}$ be increasing functions. Then for $m < t < n$, $m, n \in [a,b]$, the kernel $K_m^n(E_{\mu,\alpha',l}^{\gamma,\delta,k,c}, g; \phi)$ satisfies the following inequality:

$$K_t^m(E_{\mu,\alpha',l}^{\gamma,\delta,k,c}, g; \phi) g'(t) \leq K_n^m(E_{\mu,\alpha',l}^{\gamma,\delta,k,c}, g; \phi) g'(t). \tag{1.14}$$

This can be obtained from following two straightforward inequalities:

$$\frac{\phi(g(t)-g(m))}{g(t)-g(m)} g'(t) \leq \frac{\phi(g(n)-g(m))}{g(n)-g(m)} g'(t), \tag{1.15}$$

$$E_{\mu,\alpha',l}^{\gamma,\delta,k,c}(\omega(g(t)-g(m))^{\mu}; p) \leq E_{\mu,\alpha',l}^{\gamma,\delta,k,c}(\omega(g(n)-g(m))^{\mu}; p). \tag{1.16}$$

The reverse of inequality (1.14) holds when g and $\frac{\phi}{I}$ are of opposite monotonicity. For further properties see [8].

The aim of this paper is to obtain inequalities in compact form for unified integral operators by using convexity of composition of two functions. These inequalities investigate further results for several known integral operators. In Section 2, upper bounds of unified integral operators (1.9) and (1.10) are established by using composite convex functions. Further by using an additional condition of symmetry, two sided Hadamard type bounds are obtained. Moreover some bounds are studied by using convexity of $|f'|$. Some special cases are studied in Sections 3 & 4.

2. Main Results

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a positive convex function. Let $u : J \to \mathbb{R}$, where $Rang(u) \subseteq [a,b]$ be a differentiable and strictly increasing function, g be a* *strictly increasing function. Let* $\frac{\phi}{x}$ *be an increasing function on* $[a,b]$ *and* $\alpha, l, \gamma, c \in \mathbb{R}$, $p, \mu, \delta \geq 0$ *and* $0 < k \leq \delta + \mu$. *Then for* $x \in [a,b]$ *we have*

$$\begin{aligned}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f \circ u\right)(x,\omega;p) &\leq K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) \\ &\times (g(x)-g(a))((f\circ u)(x) + (f\circ u)(a))\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f \circ u\right)(x,\omega;p) &\leq K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) \\ &\times (g(b)-g(x))((f\circ u)(x) + (f\circ u)(b))\end{aligned} \tag{2.2}$$

hence

$$\begin{aligned}&\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f \circ u\right)(x,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f \circ u\right)(x,\omega;p) \\ &\leq K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)(g(x)-g(a))((f\circ u)(x) + (f\circ u)(a)) \\ &+ K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)(g(b)-g(x))((f\circ u)(x) + (f\circ u)(b)).\end{aligned} \tag{2.3}$$

Proof. By (**P**), the following inequality holds:

$$K^t_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)g'(t) \leq K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)g'(t). \tag{2.4}$$

Using convexity of f on the identity $u(t) = \frac{u(x)-u(t)}{u(x)-u(a)}u(a) + \frac{u(t)-u(a)}{u(x)-u(a)}u(x)$ we have

$$f(u(t)) \leq \frac{u(x)-u(t)}{u(x)-u(a)}f(u(a)) + \frac{u(t)-u(a)}{u(x)-u(a)}f(u(x)). \tag{2.5}$$

The following integral inequality can be obtained from (2.4) and (2.5):

$$\begin{aligned}&\int_a^x K^t_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) f(u(t)) d(g(t)) \\ &\leq \frac{f(u(a))}{u(x)-u(a)} K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) \int_a^x (u(x)-u(t)) d(g(t)) \\ &+ \frac{f(u(x))}{u(x)-u(a)} K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) \int_a^x (u(t)-u(a)) d(g(t)) \\ &= \frac{f(u(a))}{u(x)-u(a)} K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) u(x) \int_a^x d(g(t)) \\ &- \frac{f(u(a))}{u(x)-u(a)} K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) \int_a^x u(t) d(g(t)) \\ &+ \frac{f(u(x))}{u(x)-u(a)} K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) \int_a^x u(t) d(g(t)) \\ &- \frac{f(u(x))}{u(x)-u(a)} K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi) u(a) \int_a^x d(g(t)).\end{aligned}$$

In the last expression the second term is purely negative due to the conditions are defined on functions f, u and g, and third term is purely positive. Therefore the following inequality holds:

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\circ u\right)(x,\omega;p) \le K_x^a(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi)(g(x)-g(a)) \\ ((f\circ u)(x)+(f\circ u)(a)). \tag{2.6}$$

Again, by (**P**), the following inequality holds:

$$K_t^x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi)g'(t) \le K_b^x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi)g'(t). \tag{2.7}$$

On the other hand using convexity of f on the identity $u(t)=\frac{u(t)-u(x)}{u(b)-u(x)}u(b)+\frac{u(b)-u(t)}{u(b)-u(x)}u(x)$ we have

$$f(u(t)) \le \frac{u(t)-u(x)}{u(b)-u(x)}f(u(b)) + \frac{u(b)-u(t)}{u(b)-u(x)}f(u(x)). \tag{2.8}$$

The following integral inequality is obtained from (2.7) and (2.8):

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\circ u\right)(x,\omega;p) \le K_b^x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi) \\ (g(b)-g(x))((f\circ u)(x)+(f\circ u)(b)). \tag{2.9}$$

By adding (2.6) and (2.9), (2.3) is obtained.

□

Remark 1. By setting $u(x)=x$, [15, Theorem 8] can be obtained.

Corollary 1. *Under the suppositions of Theorem 1, if $(f\circ u)\in L_\infty[a,b]$, then the following inequalities hold:*

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\circ g\right)(x,\omega;p)\right| \le K\|f\circ u\|_\infty, \tag{2.10}$$

and

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\circ g\right)(x,\omega;p)\right| \le K\|f\circ u\|_\infty, \tag{2.11}$$

where $K=(g(b)-g(a))K_b^a(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi)$.

Proof. From (2.1) we have

$$\left|\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\circ u\right)(x,\omega;p)\right| \le K_b^a(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g;\phi) \\ \times (g(b)-g(a))\|f\circ u\|_\infty \tag{2.12}$$

that is (2.10) holds. Similarly, (2.2) gives (2.11). □

We need the following lemma to prove the upcoming theorem:

Lemma 3. *Let $f: I\to\mathbb{R}$ be a convex and increasing function and let $u: J\to\mathbb{R}$, $Rang(u)\subseteq I$ be convex and symmetric about $\frac{a+b}{2}$ for $a,b\in J$. Then we have*

$$(f\circ u)\left(\frac{a+b}{2}\right) \le (f\circ u)(x) \tag{2.13}$$

for all $x\in[a,b]$.

Proof. Since f and u are convex functions moreover f is increasing, therefore $f\circ u$ is convex. Also u is symmetric about $\frac{a+b}{2}$ so is $f\circ u$. Hence applying Lemma 2 we get (2.13) □

The upcoming theorem provides the Hadamard type estimation of integral operators (1.9) and (1.10).

Theorem 2. *With the assumptions of Theorem 1, in addition if* u *is symmetric about* $\frac{a+b}{2}$, *then the following inequality holds:*

$$(f\circ u)\left(\frac{a+b}{2}\right)\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a,\omega;p)\right. \tag{2.14}$$
$$\left.+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b,\omega;p)\right)\leq\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\circ u\right)(a,\omega;p)$$
$$+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\circ u\right)(b,\omega;p)\leq 2(g(b)-g(a))$$
$$\times K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)\left((f\circ u)(a)+(f\circ u)(b)\right).$$

Proof. By (**P**), the following inequality holds:

$$K^t_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x)\leq K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x). \tag{2.15}$$

Using convexity of f on the identity $u(t)=\frac{u(x)-u(a)}{u(b)-u(a)}u(b)$ $+\frac{u(b)-u(x)}{u(b)-u(a)}u(a)$ we have

$$f(u(x))\leq\frac{u(x)-u(a)}{u(b)-u(a)}f(u(b))+\frac{u(b)-u(x)}{u(b)-u(a)}f(u(a)). \tag{2.16}$$

The following integral inequality can be obtained from (2.15) and (2.16):

$$\int_a^b K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)f(u(x))d(g(x))$$
$$\leq\frac{f(u(b))}{u(b)-u(a)}K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)\int_a^b(u(x)-u(a))d(g(x))$$
$$+\frac{f(u(a))}{u(b)-u(a)}K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)\int_a^b(u(b)-u(x))d(g(x)).$$

Integrating by parts the above inequality gives

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\circ u\right)(b,\omega;p)\leq K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi) \tag{2.17}$$
$$(g(b)-g(a))((f\circ u)(a)+(f\circ u)(b)).$$

Again, by (**P**), the following inequality holds:

$$K^x_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x)\leq K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x). \tag{2.18}$$

The following inequality can be obtained from (2.16) and (2.18):

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\circ u\right)(a,\omega;p)\leq K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi) \tag{2.19}$$
$$(g(b)-g(a))((f\circ u)(a)+(f\circ u)(b)).$$

By adding (2.17) and (2.19), we have

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\circ u\right)(b;p)+\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\circ u\right)(a;p) \tag{2.20}$$
$$\leq 2(g(b)-g(a))K^a_b(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)((f\circ u)(a)$$
$$+(f\circ u)(b)).$$

Multiplying both sides of (2.13) by $g'(x)K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)$, then integrating over $[a, b]$ we get

$$(2.21)\qquad (f \circ u)\left(\frac{a+b}{2}\right)\int_a^b K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)d(g(x)) \\ \leq \int_a^b K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(f \circ u)(x)d(g(x)).$$

By using (1.10) of Definition 4 we get

$$(2.22)\qquad (f \circ u)\left(\frac{a+b}{2}\right)\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}1\right)(a, \omega; p) \leq \left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f \circ u\right)(a, \omega; p).$$

Multiplying both sides of (2.13) by $K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)g'(x)$ and integrating over $[a, b]$ we get

$$(2.23)\qquad (f \circ u)\left(\frac{a+b}{2}\right)\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}1\right)(b, \omega; p) \leq \left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f \circ u\right)(b, \omega; p),$$

by adding (2.22) and (2.23), the following inequality is obtained:

$$(2.24)\qquad (f \circ u)\left(\frac{a+b}{2}\right)\left(\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}1\right)(a, \omega; p)+\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}1\right)(b, \omega; p)\right) \\ \leq \left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f \circ u\right)(a, \omega; p) + \left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f \circ u\right)(b, \omega; p).$$

From (2.20) and (2.24), inequality (2.14) is obtained. □

Remark 2. By setting $u(x) = x$, [15, Theorem 22] can be obtained.

Theorem 3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. If $|f'|$ is convex. Let $u : J \to \mathbb{R}$, where $Rang(u) \subseteq [a, b]$ be a differentiable and strictly increasing function, g is also a strictly increasing function. Let $\frac{\phi}{x}$ be an increasing function and $\alpha, l, \gamma, c \in \mathbb{R}$, $p, \mu, \delta \geq 0$ and $0 < k \leq \delta + \mu$. Then for $x \in (a, b)$ we have*

$$(2.25)\qquad \left|\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(f' \circ u)\right)(x, \omega; p) + \left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}(f' \circ u)\right)(x, \omega; p)\right| \leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \\ \times (g(x) - g(a))(|(f' \circ u)(x)| + |(f' \circ u)(a)|) \\ + K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi)(g(b) - g(x))(|(f' \circ u)(x)| + |(f' \circ u)(b)|).$$

Proof. Using convexity of $|f'|$ on the identity
$u(t) = \frac{u(x)-u(t)}{u(x)-u(a)}u(a) + \frac{u(t)-u(a)}{u(x)-u(a)}u(x)$, $x \in (a, b)$
we have

$$(2.26)\qquad |f'(u(t))| \leq \frac{u(x) - u(t)}{u(x) - u(a)}|f'(u(a))| + \frac{u(t) - u(a)}{u(x) - u(a)}|f'(u(x))|,$$

from which we can write

$$(2.27)\qquad -\left(\frac{u(x) - u(t)}{u(x) - u(a)}|f'(u(a))| + \frac{u(t) - u(a)}{u(x) - u(a)}|f'(u(x))|\right) \\ \leq f'(u(t)) \leq \left(\frac{u(x) - u(t)}{u(x) - u(a)}|f'(u(a))| + \frac{u(t) - u(a)}{u(x) - u(a)}|f'(u(x))|\right).$$

Let first we consider the right hand side inequality of the above inequality i.e.

$$f'(u(t)) \leq \frac{u(x)-u(t)}{u(x)-u(a)}|f'(u(a))| + \frac{u(t)-u(a)}{u(x)-u(a)}|f'(u(x))|. \tag{2.28}$$

The following integral inequality can be obtained from (2.4) and (2.28):

$$\begin{aligned}&\int_a^x K_x^t(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) f'(u(t)) d(g(t)) \\ &\leq \frac{|f'(u(a))|}{u(x)-u(a)} K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \int_a^x (u(x)-u(t)) d(g(t)) \\ &+ \frac{f'(u(x))|}{u(x)-u(a)} K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \int_a^x (u(t)-u(x)) d(g(t)),\end{aligned}$$

which gives

$$\begin{aligned}\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(f' \circ u)\right)(x,\omega;p) &\leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \\ &\times (g(x)-g(a))(|(f' \circ u)(x)| + |(f' \circ u)(a)|).\end{aligned} \tag{2.29}$$

Now we consider the left hand side inequality from the inequality (2.27) and proceed as we did for the right hand side inequality we have

$$\begin{aligned}\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(f' \circ u)\right)(x,\omega;p) &\geq -K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \\ &\times (g(x)-g(a))(|(f' \circ u)(x)| + |(f' \circ u)(a)|).\end{aligned} \tag{2.30}$$

From (2.29) and (2.30), the following inequality is obtained:

$$\begin{aligned}\left|\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(f' \circ u)\right)(x,\omega;p)\right| &\leq K_x^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \\ &\times (g(x)-g(a))(|(f' \circ u)(x)| + |(f' \circ u)(a)|).\end{aligned} \tag{2.31}$$

On the other hand using convexity of $|f'(t)|$ on the identity $u(t) = \frac{u(t)-u(x)}{u(b)-u(x)} u(b) + \frac{u(b)-u(t)}{u(b)-u(x)} u(x)$ we have

$$|f'(u(t))| \leq \frac{u(t)-u(x)}{u(b)-u(x)}|f'(u(b))| + \frac{u(b)-u(t)}{u(b)-u(x)}|f'(u(x))|. \tag{2.32}$$

The following integral inequality can be obtained from (2.7) and (2.32):

$$\begin{aligned}\left|\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}(f' \circ u)\right)(x,\omega;p)\right| &\leq K_b^x(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi) \\ &\times (g(b)-g(x))(|(f' \circ u)(x)| + |(f' \circ u)(b)|).\end{aligned} \tag{2.33}$$

From (2.31) and (2.33), inequality (2.25) is obtained. □

Remark 3. By setting $u(x) = x$, [15, Theorem 25] can be obtained.

3. Hadamard type inequalities

In this section some interesting implications of Theorem 2 are obtained by setting specific values of the functions ϕ and g in (2.14). These results actually give different versions of Hadamard inequality for fractional and conformable integrals deduced in [15, Remark 6] .

COROLLARY 2. *If we consider $\phi(t)=\frac{\Gamma(\alpha)t^{\alpha/k}}{k\Gamma_k(\alpha)}$ and $p=\omega=0$, then (2.14) satisfies the following Hadamard type inequality for generalized Riemann-Liouville fractional integral operators for $\alpha\geq k$ defined in [14]:*

$$\begin{aligned}
&\frac{2}{\alpha\Gamma_K(\alpha)}(f\circ u)\left(\frac{a+b}{2}\right)(g(b)-g(a))^{\alpha/k}\\
&\leq {}_g^{\alpha}I_{a^+}^{k}(f\circ u)(b)+_g^{\alpha}I_{b^-}^{k}(f\circ u)(a)\\
&\leq \frac{2}{k\Gamma_k(\alpha)}(g(b)-g(a))^{\alpha/k}((f\circ u)(a)+(f\circ u)(b)).
\end{aligned}$$

COROLLARY 3. *If we consider $\phi(t)=t^{\alpha}$ and $g(x)=I(x)=x$ with $p=\omega=0$, then (2.14) satisfies the following Hadamard type inequality for Riemann-Liouville integral operators defined in [13]:*

$$\begin{aligned}
&\frac{2}{\alpha\Gamma(\alpha)}(f\circ u)\left(\frac{a+b}{2}\right)(b-a)^{\alpha}\\
&\leq {}^{\alpha}I_{a^+}(f\circ u)(b)+{}^{\alpha}I_{b^-}(f\circ u)(a)\\
&\leq \frac{2}{\Gamma(\alpha)}(b-a)^{\alpha}((f\circ u)(a)+(f\circ u)(b)).
\end{aligned}$$

COROLLARY 4. *If we consider $\phi(t)=\frac{t^{\alpha/k}\Gamma(\alpha)}{k\Gamma_k(\alpha)}$ and $g(x)=I(x)=x$, $p=\omega=0$, then (2.14) satisfies the following Hadamard type inequality for k-fractional Riemann-Liouville integral operators along with $\alpha\geq k$ defined in [17]:*

$$\begin{aligned}
&\frac{2}{\alpha\Gamma_k(\alpha)}(f\circ u)\left(\frac{a+b}{2}\right)(g-a)^{\alpha/k}\\
&\leq {}^{\alpha}I_{a^+}^{k}(f\circ u)(b)+^{\alpha}I_{b^-}^{k}(f\circ u)(a)\\
&\leq \frac{2}{k\Gamma_k(\alpha)}(g-a)^{\alpha/k}((f\circ u)(a)+(f\circ u)(b)).
\end{aligned}$$

COROLLARY 5. *If we consider $\phi(t)=t^{\alpha},\alpha>0$ and $g(x)=\frac{x^{\rho}}{\rho}$, $\rho>0$ with $p=\omega=0$, then (2.14) satisfies the following Hadamard type inequality for Katugampola fractional integral operators defined in [2]:*

$$\begin{aligned}
&\frac{2}{\rho^{\alpha}\alpha\Gamma(\alpha)}(b^{\rho}-a^{\rho})^{\alpha}(f\circ u)\left(\frac{a+b}{2}\right)\\
&\leq {}^{\rho}I_{a^+}^{\alpha}(f\circ u)(b)+^{\rho}I_{b^-}^{\alpha}(f\circ u)(a)\\
&\leq \frac{2}{\rho^{\alpha}\Gamma(\alpha)}(b^{\rho}-a^{\rho})^{\alpha}((f\circ u)(a)+(f\circ u)(b)).
\end{aligned}$$

COROLLARY 6. *If we consider $\phi(t)=t^{\alpha},\alpha>0$ and $g(x)=\frac{x^{s+1}}{s+1}$, $s>0, p=\omega=0$, then (2.14) satisfies the following Hadamard type inequality for conformable integral operators:*

$$\begin{aligned}
&\frac{2}{(s+1)^{\alpha}\alpha\Gamma(\alpha)}(b^{s+1}-a^{s+1})^{\alpha}(f\circ u)\left(\frac{a+b}{2}\right)\\
&\leq {}^{s}I_{a^+}^{\alpha}(f\circ u)(b)+^{s}I_{b^-}^{\alpha}(f\circ u)(a)\\
&\leq \frac{2}{(s+1)^{\alpha}\Gamma(\alpha)}(b^{s+1}-a^{s+1})^{\alpha}((f\circ u)(a)+(f\circ u)(b)).
\end{aligned}$$

Corollary 7. *If we consider $\phi(t) = \frac{t^{\alpha/k}\Gamma(\alpha)}{k\Gamma_k(\alpha)}$ and $g(x) = \frac{x^{s+1}}{s+1}$, $s > 0$, $p = \omega = 0$, then (2.14) satisfies the following Hadamard type inequality for conformable integral operators with $\alpha \geq k$ defined in [26]:*

$$\frac{2(b^{s+1} - a^{s+1})^{\alpha/k}}{(s+1)^{\alpha/k}\alpha\Gamma_k(\alpha)}(f \circ u)\left(\frac{a+b}{2}\right) \leq {}^{s}_{k}I^{\alpha}_{a^+}(f \circ u)(b)$$
$$+ {}^{s}_{k}I^{\alpha}_{b^-}(f \circ u)(a) \leq \frac{2}{(s+1)^{\alpha/k}k\Gamma(\alpha)}(b^{s+1} - a^{s+1})^{\alpha/k}((f \circ u)(a) + (f \circ u)(b)).$$

Corollary 8. *If we consider $\phi(t) = t^{\alpha}$ and $g(x) = \frac{x^{\beta+s}}{\beta+s}$, $\beta, s > 0$, $p = \omega = 0$, then (2.14) satisfies the following Hadamard type inequality for generalized conformable integral operators defined in [12]:*

$$\frac{2}{(\beta+s)^{\alpha}\alpha\Gamma(\alpha)}(b^{\beta+} - a^{\beta+s})^{\alpha}(f \circ u)\left(\frac{a+b}{2}\right)$$
$$\leq {}^{s}_{\beta}I^{\alpha}_{a^+}(f \circ u)(b) + {}^{s}_{\beta}I^{\alpha}_{b^-}(f \circ u)(a)$$
$$\leq \frac{2}{(\beta+s)^{\alpha}\Gamma(\alpha)}(b^{\beta+s} - a^{\beta+s})^{\alpha}((f \circ u)(a) + (f \circ u)(b)).$$

4. Further implications

In this section we apply Theorem 1 to get the boundedness and continuity of integral operators (1.9)and (1.10) from which reader can reproduce boundedness of integral operators given in [15, Remark 6 & 7]. Moreover we present some examples which provide upper bounds of Riemann-Liouville fractional integrals.

Theorem 4. *If $f \in L_{\infty}[a,b]$, then integral operators defined in (1.9) and (1.10) are continuous.*

Proof. Let $u(x) = x$, then u is strictly increasing, so (2.1) becomes

$$\left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p) \leq K^{a}_{x}(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)(g(x) - g(a))(f(x) + f(a)). \tag{4.1}$$

Using (2.2) we have

$$\left|\left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p)\right| \leq 2K^{a}_{b}(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)(g(b) - g(a))\|f\|_{\infty}. \tag{4.2}$$

$\left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(x,\omega;p)$ is bounded, also it is linear and hence continuous. Similarly, continuity of $\left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(x,\omega;p)$ can be proved. □

Corollary 9. *Under the assumptions of Theorem 1 the following inequality holds:*

$$I^{\alpha}_{b^-}u(x) + I^{\alpha}_{a^+}u(x) \leq \frac{1}{\Gamma(\alpha)}\left((x-a)^{\alpha}(u(x) - u(a)) + (b-x)^{\alpha}(u(b) - u(x))\right). \tag{4.3}$$

Proof. Let $f(x) = x$. Then f is convex and from (2.1) we have

$$\left({}_{g}F^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}u(x)\right)(x,\omega;p) \leq K^{a}_{x}(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)(u(x) + u(a))$$
$$= \phi(g(x) - g(a))(u(x) + u(a))E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^{\mu}; p).$$

Further if $\phi(t) = t^\alpha$ and $g(x) = x$, then we have

$$\sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{\omega^n}{(l)_{n\delta}} \int_a^x (x - t)^{\mu n + \alpha - 1} u(t) dt$$
$$\leq (x - a)^\alpha (u(x) - u(a)) E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega((x - a)^\mu; p),$$

which gives

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega; p) I^{\mu n + \alpha}_{a^+} u(x) \leq (x - a)^\alpha (u(x) - u(a)) E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega((x - a)^\mu; p).$$

By setting $\omega = p = 0$ we get

$$I^\alpha_{a^+} u(x) \leq \frac{1}{\Gamma(\alpha)} (x - a)^\alpha (u(x) - u(a)). \tag{4.4}$$

Similarly, one can also obtain

$$I^\alpha_{b^-} u(x) \leq \frac{1}{\Gamma(\alpha)} (b - x)^\alpha (u(b) - u(x)). \tag{4.5}$$

From (4.4) and (4.5), (4.3) is obtained. □

Corollary 10. *Under the assumptions of Theorem 3 the following inequality holds:*

(4.6)
$$I^\alpha_{a^+} |u|(x) + I^\alpha_{b^-} |u|(x) \leq \frac{1}{\Gamma(\alpha)} \left((x - a)^\alpha (|u|(x) - |u|(a)) + (b - x)^\alpha (|u|(b) - |u|(x)) \right).$$

Proof. Let $f(x) = \dfrac{x^2}{2}$. Then $|f'(x)| = |x|$ is convex and from (2.29) we have

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} |u|(x) \right)(x, \omega; p) \leq K^a_x(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g; \phi)(|u|(x) + |u|(a))$$
$$= \phi(g(x) - g(a))(|u|(x) + |u|(a)) E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(a))^\mu; p).$$

Further if $\phi(t) = t^\alpha$ and $g(x) = x$, then we have

$$\sum_{n=0}^{\infty} \frac{\beta_p(\gamma + nk, c - \gamma)}{\beta(\gamma, c - \gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{\omega^n}{(l)_{n\delta}}$$
$$\times \int_a^x (x - t)^{\mu n + \alpha - 1} |u(t)| dt \leq (x - a)^\alpha (|u|(x) - |u|(a))$$
$$\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega((x - a)^\mu; p),$$

which gives

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega; p) I^{\mu n + \alpha}_{a^+} |u|(x) \leq (x - a)^\alpha (|u|(x) - |u|(a))$$
$$\times E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega((x - a)^\mu; p),$$

by setting $\omega = p = 0$ we get

$$I^\alpha_{a^+} |u|(x) \leq \frac{1}{\Gamma(\alpha)} (x - a)^\alpha (|u|(x) - |u|(a)). \tag{4.7}$$

Similarly, one can also obtain

$$I^\alpha_{b^-} |u|(x) \leq \frac{1}{\Gamma(\alpha)} (b - x)^\alpha (|u|(b) - |u|(x)). \tag{4.8}$$

From (4.7) and (4.8), (4.6) is obtained. □

5. Concluding remarks

This paper studies a unified integral operator via convex composition of two functions. In the consequent generalized bounds of this operator are derived in a compact form which the bounds of several kinds of fractional and conformable integral operators can be produced. The fractional and conformable integral operators defined in [2, 4, 7, 9, 10, 12, 17, 18, 20, 24, 29, 30] are directly linked and satisfy all the results. Furthermore various versions of Hadamard inequality are produced and boundedness of all the considered operators is deduced. Also upper bounds of Riemann-Liouville fractional integrals are established in two corollaries while reader can obtain such bounds for other fractional and conformable integrals.

Competing interests: The authors do not have any competing interests.
Authors contributions: All authors have contributed equally in this paper.

References

[1] M. Andrić, G. Farid and J. Pečarić, *A further extension of Mittag-Leffler function*, Fract. Calc. Appl. Anal. **21** (5) (2018), 1377–1395.

[2] H. Chen and U. N. Katugampola, *Hermite-Hadamard and Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals*, J. Math. Anal. Appl. **446** (2017), 1274–1291.

[3] C. Cesarano, *Generalized special functions in the description of fractional diffusive equations*, CAIM **10** (2019), 31–40

[4] S. S. Dragomir, *Inequalities of Jensens type for generalized k-g-fractional integrals of functions for which the composite* $f \circ g^{-1}$ *is convex*, Fract. Differ. Calc. **8** (1) (2018), 127–150.

[5] R. Dubey, N. L. Mishra and C. Cesarano, *Multi objective fractional symmetric duality in mathematical programming with (C,Gf)-invexity assumptions*, AXIOMS **8** (2019), 1–11.

[6] G. Farid, *Some Riemann-Liouville fractional integrals inequalities for convex function*, J. Anal. **27** (4) (2019), 1095–1102.

[7] G. Farid, *Existence of an integral operator and its consequences in fractional and conformable integrals*, Open J. Math. Sci. **3** (3) (2019), 210–216.

[8] Z. He, G. Farid, A. U. Haq and K. Mahreen, *Bounds of a unified integral operator for* (s, m)*-convex functions and their consequences*, AIMS Mathematics **5** (6) (2020), 5510–5520.

[9] S. Habib, S. Mubeen and M. N. Naeem, *Chebyshev type integral inequalities for generalized k-fractional conformable integrals*, J. Inequal. Spec. Funct. **9** (4) (2018), 53–65.

[10] F. Jarad, E. Ugurlu, T. Abdeljawad and D. Baleanu, *On a new class of fractional operators*, Adv. Difference Equ. (2017), 2017:247.

[11] S. Kermausuor, *Simpson's type inequalities for strongly* (s, m)*-convex functions in the second sense and applications*, Open J. Math. Sci. **3** (1) (2019), 74–83.

[12] T. U. Khan and M. A. Khan, *Generalized conformable fractional operators*, J. Comput. Appl. Math. **346** (2019), 378–389.

[13] A. A. Kilbaş, H. M. Srivastava and J. J Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies, 204, Elsevier, New York-London, 2006.

[14] Y. C. Kwun, G. Farid, W. Nazeer, S. Ullah and S. M. Kang, *Generalized Riemann-Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities*, IEEE Access **6** (2018), 64946–64953.

[15] Y. C. Kwun, G. Farid, S. Ullah, W. Nazeer, K. Mahreen and S. M. Kang, *Inequalities for a unified integral operator and associated results in fractional calculus,* IEEE Access **7** (2019), 126283–126292.

[16] S. Mubeen and A. Rehman, *A note on k-Gamma function and Pochhammer k-symbol*, J. Math. Sci. **6** (2) (2014), 93–107.

[17] S. Mubeen and G. M. Habibullah, *k-fractional integrals and applications*, Int. J. Contemp. Math. **7** (2) (2012), 89–94.

[18] T. R. Parbhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J. **19** (1971), 7–15.

[19] S. I. Butt, M. Nadeem and G. Farid,*On Caputo fractional derivatives via exponential s?convex functions*, Turkish Journal of Science **5** (2) (2020), 140–146.

[20] G. Rahman, D. Baleanu, M. A. Qurashi, S. D. Purohit, S. Mubeen and M. Arshad, *The extended Mittag-Leffeler function via fractional calculus*, J. Nonlinear Sci. Appl. **10** (2013), 4244–4253.

[21] G. Rahman, A. Khan, T. Abdeljwad and K. S. Nisar, The Minkowski inequalities via generalized proportional fractional integral operators, Adv. Difference equ. (2019) 2019:287.

[22] S. Rashid, T. Abdeljawad, F. Jarad and M. A. Noor, *Some Estimates for Generalized Riemann-Liouville fractional integrals of exponentially convex functions and their applications*, Mathematics **7** (9) (2019), 807.

[23] A. W. Roberts and D. E. Varberg, *Convex functions*, Acadamic press New York and London 1993.

[24] T. O. Salim, and A. W. Faraj, *A generalization of Mittag-Leffler function and integral operator associated with integral calculus*, J. Fract. Calc. Appl. **3** (5) (2012), 1–13.

[25] M. Z. Sarıkaya and N. Alp, *On Hermite-Hadamard-Fejér type integral inequalities for generalized convex functions via local fractional integrals*, Open J. Math. Sci. **3** (1) (2019), 273–284.

[26] M. Z. Sarıkaya, M. Dahmani, M. E. Kiriş and F. Ahmad, (k,s)*-Riemann-Liouville fractional integral and applications*, Hacet. J. Math. Stat. **45** (1) (2016), 77–89. doi:10.15672/HJMS.20164512484

[27] E. Set, M. A. Noor and M. U. Awan, *A. Gözpinar, Generalized Hermite-Hadamard type inequalities involving fractional integral operators*, J. Inequal. Appl., (2017), 2017:169.

[28] E. Set, J. Choi and B. Çelik, *Certain Hermite-Hadamard type inequalities involving generalized fractional integral operators*, RACSAM **112** (4) (2018), 1539–1547.

[29] H. M. Srivastava and Z. Tomovski, *Fractional calculus with an integral operator containing generalized Mittag-Leffler function in the kernel*, Appl. Math. Comput. **211** (1) (2009), 198–210.

[30] T. Tunç, H. Budak, F. Usta and M. Z. Sarıkaya, *On new generalized fractional integral operators and related fractional inequalities*, https://www.researchgate.net/publication/313650587.

Lakshmi Narayan Mishra
Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology (VIT) University,
Vellore 632 014, Tamil Nadu, India
E-mail: lakshminarayanmishra04@gmail.com

Ghulam Farid
Department of Mathematics, COMSATS University Islamabad,
Attock, Pakistan
E-mail: faridphdsms@hotmail.com, ghlmfarid@ciit-attock.edu.pk

Kahkashan Mahreen
Department of Mathematics, COMSATS University Islamabad,
Attock, Pakistan
E-mail: kahkashan.mpa@gmail.com