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EXISTENCE OF GENERALISED LOGARITHMIC PROXIMATE ORDER AND GENERALISED LOGARITHMIC PROXIMATE TYPE OF AN ENTIRE FUNCTION

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Abstract. In this paper we introduce generalised logarithmic proximate order, generalised logarithmic proximate type of an entire function and prove the corresponding existence theorems. Also we investigate some theorems on the application of generalised logarithmic proximate order.

1. Introduction

Let $f(z)$ be an entire function defined in the finite complex plane $\mathbb{C}$. The maximum modulus function corresponding to entire function $f(z)$ is defined as $M_f(r) = \sup_{|z|=r} |f(z)|$. In 1963, Sato [4] introduced the definition of generalised order and lower order of $f(z)$ as

$$\begin{aligned} \rho_k &= \limsup_{r\to\infty} \frac{\log^k M_f(r)}{\log r} = \limsup_{r\to\infty} \frac{\log^{k-1} T_f(r)}{\log r}, \\ \lambda_k &= \liminf_{r\to\infty} \frac{\log^k M_f(r)}{\log r} = \liminf_{r\to\infty} \frac{\log^{k-1} T_f(r)}{\log r} \end{aligned}$$

respectively where $T_f(r)$ is the Nevanlinna characteristic function of $f(z)$. There are two other indicators of growth of an entire function $f(z)$,the generalised type T_k and the generalised lower type t_k. They are defined for all ρ_k, $0 < \rho_k < \infty$ as

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^{k-1} M_f(r)}{r^{\rho_k}} &= T_k, \\ \liminf_{r\to\infty} \frac{\log^{k-1} M_f(r)}{r^{\rho_k}} &= t_k. \end{aligned}$$

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Now the logarithmic order of $f(z)$ be defined by [1]

$$\rho_{\log}=\limsup_{r\to\infty}\frac{\log^{+}\log^{+}M_f\left(r\right)}{\log\log r}.$$

Since [2]

$$T_f\left(r\right)\leq\log^{+}M_f\left(r\right)\leq\left(\frac{R+r}{R-r}\right)T_f\left(R\right)\tag{1}$$

for $0<r<R$, $T_f\left(r\right)$ and $\log^{+}M_f\left(r\right)$ are of the same logarithmic order. Hence

$$\rho_{\log}=\limsup_{r\to\infty}\frac{\log^{+}T_f\left(r\right)}{\log\log r}.$$

Throughout this paper we use the following notations [7]

$$\log^{[0]}x=x,\ \ \log^{k}x=\log\left(\log^{k-1}x\right);\ k=1,2,3,...$$

and

$$\exp^{[0]}x=x,\ \ \exp^{k}x=\exp\left(\exp^{k-1}x\right);\ k=1,2,3,...\ .$$

One can write the following definitions as:

Definition 1.1. If $f(z)$ is an entire function, the generalised logarithmic order and the generalised logarithmic lower order of $f(z)$ are defined by

$$\begin{aligned}\rho_{\log}^{k}&=\limsup_{r\to\infty}\frac{\log^{k}M_f\left(r\right)}{\log\log r}\\ \lambda_{\log}^{k}&=\liminf_{r\to\infty}\frac{\log^{k}M_f\left(r\right)}{\log\log r}\end{aligned}$$

respectively, where $k\geq 2$ is an integer.

If $f(z)$ is a meromorphic function, the generalised logarithmic order and the generalised logarithmic lower order of $f(z)$ are defined by

$$\begin{aligned}\rho_{\log}^{k}&=\limsup_{r\to\infty}\frac{\log^{k-1}T_f\left(r\right)}{\log\log r},\\ \lambda_{\log}^{k}&=\liminf_{r\to\infty}\frac{\log^{k-1}T_f\left(r\right)}{\log\log r}\end{aligned}$$

respectively, where $k\geq 2$ is an integer.

Definition 1.2. The generalised logarithmic type and the generalised logarithmic lower type of an entire function $f(z)$ are defined as,

$$\begin{aligned}T_{\log}^{k}&=\limsup_{r\to\infty}\frac{\log^{k-1}M_f\left(r\right)}{\left(\log r\right)^{\rho_{\log}^{k}}},\\ t_{\log}^{k}&=\liminf_{r\to\infty}\frac{\log^{k-1}M_f\left(r\right)}{\left(\log r\right)^{\rho_{\log}^{k}}}\end{aligned}$$

respectively, where $k\geq 2$ is an integer and $0<\rho_{\log}^{k}<\infty$.

The generalised logarithmic type and the generalised logarithmic lower type of a meromorphic function $f(z)$ are defined as,

$$T_{\log}^{k}\left(t_{\log}^{k}\right)=\limsup_{r\to\infty}\left(\liminf_{r\to\infty}\right)\frac{\log^{k-2}T_f\left(r\right)}{\left(\log r\right)^{\rho_{\log}^{k}}}$$

where $k \geq 2$ is an integer and $0 < \rho_{\log}^{k}$.

If $f(z)$ is an entire function of finite order ρ, it is proved (Valiron [7]) that there exists a positive continuous function $\rho(r)$ with the following properties:

(i) $\rho(r)$ is differentiable for sufficiently large values of r except at isolated points where $\rho'(r-0), \rho'(r+0)$ exist;

(ii) $\lim_{r\to\infty} \rho(r) = \rho$;

(iii) $\lim_{r\to\infty} \rho'(r) r \log r = 0$;

(iv) $\limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho(r)}} = 1$.

Such a function is called a proximate order for the entire function $f(z)$. Shah [5] gave a simple proof of the existence of proximate order of an entire function. Lahiri [3] generalised the idea for a meromorphic function.

After that Srivastava and Juneja [6] gave the proof of the existence of proximate type of an entire function as:

Definition 1.3. [6] A function $T(r)$ is said to be a proximate type of an entire function $f(z)$ of order $\rho(0 < \rho < \infty)$ and finite type T if it satisfies the following properties:

(i) $T(r)$ is real valued, continuous and piecewise differentiable for sufficiently large values of r;

(ii) $\lim_{r\to\infty} T(r) = T$;

(iii) $\lim_{r\to\infty} rT'(r) = 0$, where $T'(r)$ is either the right or the left hand derivative at points where they are different;

(iv) $\limsup_{r\to\infty} \frac{M_f(r)}{\exp\{r^{\rho} T(r)\}} = 1$.

In this paper we want to prove the existence of generalised logarithmic proximate order. Also the existence of generalised logarithmic proximate type of an entire function. We will also prove a result on the bounds of zeros and poles of a meromorphic function and further investigate on the comparative growth properties of $\log^{k-1} M_f(\exp r)$ and $\log^{k-2} T_f(\exp r)$ for an entire function $f(z)$.

2. Main Results

In this section we first introduce the definitions of generalised logarithmic proximate order and generalised logarithmic proximate type of an entire function. Then we prove their existence.

Definition 2.1. If $f(z)$ is an entire function of generalised logarithmic order $\rho_{\log}^{k}$. A function $\rho_{\log}^{k}(r)$ is said to be finite generalised logarithmic proximate order of $f(z)$ if the following properties hold:

(i) $\rho_{\log}^{k}(r)$ is differentiable for sufficiently large values of r except at isolated points where $\left(\rho_{\log}^{k}\right)'(r-0), \left(\rho_{\log}^{k}\right)'(r+0)$ exist;

(ii) $\lim_{r\to\infty} \rho_{\log}^{k}(r) = \rho_{\log}^{k}$;

(iii) $\lim_{r\to\infty} \left(\rho_{\log}^{k}\right)'(r) \prod_{i=0}^{k-1} log^{i}(r) = 0$;

(iv) $\limsup_{r\to\infty} \frac{\log^{k-1} M_f(r)}{(\log r)^{\rho^k_{\log}(r)}} = 1.$

Definition 2.2. If $f(z)$ is an entire function of generalised logarithmic order $\rho^k_{\log}$.Then the function $T^k_{\log}(r)$ is said to be generalised logarithmic proximate type of $f(z)$ of order $\rho^k_{\log}(0 < \rho^k_{\log} < \infty)$ if it satisfies the following properties:

(i) $T^k_{\log}(r)$ is differentiable for sufficiently large values of r except at isolated points where $\left(T^k_{\log}\right)'(r-0)$, $\left(T^k_{\log}\right)'(r+0)$ exist;

(ii) $\lim_{r\to\infty} T^k_{\log}(r) = T^k_{\log}$;

(iii) $\lim_{r\to\infty} \left(T^k_{\log}\right)'(r) \prod_{i=0}^{k-2} log^i(r) = 0$;

(iv) $\limsup_{r\to\infty} \frac{M_f(r)}{\exp^{k-1}\left\{(\log r)^{\rho^k_{\log}} T^k_{\log}(r)\right\}} = 1.$

Theorem 2.3. *For every entire function $f(z)$ of generalised logarithmic order $\rho^k_{\log}$, there exists a generalised logarithmic proximate order $\rho^k_{\log}(r)$.*

Proof. Let us suppose,

$$\sigma^k_{\log}(r) = \frac{\log^k M_f(r)}{\log\log r}$$

then we have

$$\limsup_{r\to\infty} \sigma^k_{\log}(r) = \rho^k_{\log}.$$

There may arise two cases:

Case I: Let $\sigma^k_{\log}(r) > \rho^k_{\log}$ for atleast a sequence of values of r tending to infinity.

Define

$$\phi^k_{\log}(r) = \max_{x\geq r} \left\{\sigma^k_{\log}(x)\right\}.$$

Therefore $\phi^k_{\log}(r)$ exists and is nonincreasing.

Let $R_1 > \exp^{k+1}(1)$ be such that $R_1 > R$ and $\sigma^k_{\log}(R) > \rho^k_{\log}$.

Then we get for $r \geq R_1 > R$,

$$\sigma^k_{\log}(r) \leq \sigma^k_{\log}(R).$$

As $\sigma^k_{\log}(r)$ is continuous, there exists $r_1 \in [R, R_1]$ such that

$$\sigma^k_{\log}(r_1) = \max_{R\leq x\leq R_1} \left\{\sigma^k_{\log}(x)\right\}.$$

Clearly $r_1 > \exp^{k+1}(1)$ and $\phi^k_{\log}(r_1) = \sigma^k_{\log}(r_1)$.

Such values r_1 will exist for a sequence of values of r tending to infinity.

Let $\rho^k_{\log}(r_1) = \phi^k_{\log}(r_1)$ and t_1 be the smallest integer not less than $1 + r_1$ such that $\phi^k_{\log}(r_1) > \phi^k_{\log}(t_1)$.

We define $\rho^k_{\log}(r) = \rho^k_{\log}(r_1)$ for $r_1 < r \leq t_1$.

Obviously $\phi^k_{\log}(r)$ and $\rho^k_{\log}(r_1) - \log^{k+1} r + \log^{k+1} t_1$ are continuous functions of r and we have

$$\lim_{r\to\infty} \rho^k_{\log}(r_1) - \log^{k+1} r + \log^{k+1} t_1 = -\infty.$$

Also, $\rho^k_{\log}(r_1) - \log^{k+1} r + \log^{k+1} t_1 > \phi^k_{\log}(t_1)$ for $r\,(> t_1)$ sufficiently close to t_1 and $\phi^k_{\log}(r)$ is nonincreasing.

Therefore one can define u_1 as

$$\begin{array}{rcl} u_1 & > & t_1 \\ \rho_{\log}^{k}(r) & = & \rho_{\log}^{k}(r_1) - \log^{k+1} r + \log^{k+1} t_1, \text{ for } t_1 \le r \le u_1 \\ \rho_{\log}^{k}(r) & = & \phi_{\log}^{k}(r), \text{ for } r = u_1. \end{array}$$

Also, we see that

$$\rho_{\log}^{k}(r) > \phi_{\log}^{k}(r), \text{ for } t_1 \le r < u_1.$$

Again let r_2 be the smallest value of r for which $r_2 \ge u_1$ and $\phi_{\log}^{k}(r_2) = \sigma_{\log}^{k}(r_2)$. If $r_2 > u_1$ then let $\rho_{\log}^{k}(r) = \phi_{\log}^{k}(r)$ for $u_1 \le r \le r_2$.

Note that $\phi_{\log}^{k}(r)$ is constant in $u_1 \le r \le r_2$. Then $\rho_{\log}^{k}(r)$ is constant in $u_1 \le r \le r_2$.

Continuing this process infinitely and we obtain that $\rho_{\log}^{k}(r)$ is differentiable in adjacent intervals.

Also, $\left(\rho_{\log}^{k}\right)'(r) = 0$ or $\dfrac{-1}{\prod_{i=0}^{k} log^{i}(r)}$ and $\rho_{\log}^{k}(r) \ge \phi_{\log}^{k}(r) \ge \sigma_{\log}^{k}(r)$ for all $r \ge r_1$.

Further, $\rho_{\log}^{k}(r) = \sigma_{\log}^{k}(r)$ for a sequence of values of r tending to infinity, $\rho_{\log}^{k}(r)$ is nonincreasing for $r \ge r_1$ and

$$\begin{array}{rcl} \rho_{\log}^{k} & = & \limsup\limits_{r\to\infty} \sigma_{\log}^{k}(r) \\ & = & \lim\limits_{r\to\infty} \phi_{\log}^{k}(r). \end{array}$$

So

$$\begin{array}{rcl} \limsup\limits_{r\to\infty} \rho_{\log}^{k}(r) & = & \liminf\limits_{r\to\infty} \rho_{\log}^{k}(r) \\ & = & \lim\limits_{r\to\infty} \rho_{\log}^{k}(r) \\ & = & \rho_{\log}^{k} \end{array}$$

and

$$\lim_{r\to\infty} \left(\rho_{\log}^{k}\right)'(r) \prod_{i=0}^{k-1} log^{i}(r) = 0.$$

Further we have

$$\begin{array}{rcl} \log^{k-1} M_f(r) & = & (\log r)^{\sigma_{\log}^{k}(r)} \\ & = & (\log r)^{\rho_{\log}^{k}(r)} \end{array}$$

for a sequence of values of r tending to ∞ and

$$\log^{k-1} M_f(r) < (\log r)^{\rho_{\log}^{k}(r)}$$

for remaining r's. Therefore

$$\limsup_{r\to\infty} \frac{\log^{k-1} M_f(r)}{(\log r)^{\rho_{\log}^{k}(r)}} = 1.$$

Finally we have $\rho_{\log}^{k}(r)$ is continuous for $r \ge r_1$. It proves Case I.

Case II: Let us suppose $\sigma_{\log}^{k}(r) \le \rho_{\log}^{k}$ for all sufficiently large values of r.

In case II we have two Subcases:

Subcase A: Let $\sigma_{\log}^{k}(r)=\rho_{\log}^{k}$ for atleast a sequence of values of r tending to infinity.

We take $\rho_{\log}^{k}(r)=\rho_{\log}^{k}$ for all values of r.

Subcase B: Let $\sigma_{\log}^{k}(r)<\rho_{\log}^{k}$ for all sufficiently large values of r.

Let

$$\xi_{\log}^{k}(r)=\max_{R_2\le x\le r}\sigma_{\log}^{k}(x)$$

where $R_2>\exp^{k+1}(1)$ is such that $\sigma_{\log}^{k}(x)<\rho_{\log}^{k}$ whenever $x\ge R_2$.

Therefore $\xi_{\log}^{k}(r)$ is increasing and for all sufficiently large $x\ge R_2$,the roots of $\xi_{\log}^{k}(x)=\rho_{\log}^{k}+\log^{k+1}x-\log^{k+1}r$ are less than x.

For a suitable large value $u_2>R_2$, we define

$$\begin{aligned}\rho_{\log}^{k}(u_2) &= \rho_{\log}^{k},\\ \rho_{\log}^{k}(r) &= \rho_{\log}^{k}+\log^{k+1}r-\log^{k+1}u_2,\end{aligned}$$

for $t_2\le r\le u_2$ where $t_2<u_2$ is such that $\xi_{\log}^{k}(t_2)=\rho_{\log}^{k}(t_2)$.

In fact t_2 is given by the largest positive root of $\xi_{\log}^{k}(x)=\rho_{\log}^{k}+\log^{k+1}x-\log^{k+1}u_2$.

If $\xi_{\log}^{k}(t_2)\ne\sigma_{\log}^{k}(t_2)$, let v_1 be the upper bound of points $v\,(<t_2)$ at which $\xi_{\log}^{k}(v)=\sigma_{\log}^{k}(v)$.

Note that $\xi_{\log}^{k}(v_1)=\sigma_{\log}^{k}(v_1)$.

We define

$$\rho_{\log}^{k}(r)=\xi_{\log}^{k}(r)$$

for $v_1\le r\le t_2$.

One can check that $\xi_{\log}^{k}(r)$ is constant in $v_1\le r\le t_2$. Thus $\rho_{\log}^{k}(r)$ is constant in $[v_1,t_2]$.

If $\xi_{\log}^{k}(t_2)=\sigma_{\log}^{k}(t_2)$, we take $v_1=t_2$.

We choose $u_3>u_2$ suitably large and let

$$\begin{aligned}\rho_{\log}^{k}(u_2) &= \rho_{\log}^{k},\\ \rho_{\log}^{k}(r) &= \rho_{\log}^{k}+\log^{k+1}r-\log^{k+1}u_3,\end{aligned}$$

for $t_3\le r\le u_3$ where $t_3<u_3$ is such that $\xi_{\log}^{k}(t_3)=\rho_{\log}^{k}(t_3)$.

If $\xi_{\log}^{k}(t_3)\ne\rho_{\log}^{k}(t_3)$, let $\rho_{\log}^{k}(r)=\xi_{\log}^{k}(r)$ for $v_2\le r\le t_3$, where v_2 has a similar property as that of v_1.

Similarly as before $\rho_{\log}^{k}(r)$ is constant in $[v_2,t_3]$.

If $\xi_{\log}^{k}(t_3)=\sigma_{\log}^{k}(t_3)$, we take $v_2=t_3$.

Let

$$\rho_{\log}^{k}(r)=\rho_{\log}^{k}(v_2)+\log^{k+1}v_2-\log^{k+1}r$$

for $t_4\le r\le v_2$ where $t_4\,(<v_2)$ is the point of intersection of $y=\rho_{\log}^{k}$ and $y=\rho_{\log}^{k}(v_2)+\log^{k+1}v_2-\log^{k+1}x$.

We can choose u_3 so large that $u_2<t_4$.

Let $\rho_{\log}^{k}(r)=\rho_{\log}^{k}$ for $u_2\le r\le t_4$.

We repeat this process.

Now we have for all $r\ge u_2,\rho_{\log}^{k}\ge\rho_{\log}^{k}(r)\ge\xi_{\log}^{k}(r)\ge\sigma_{\log}^{k}(r)$ and $\rho_{\log}^{k}(r)=\sigma_{\log}^{k}(r)$ for $r=v_1,v_2,....$

So we get

$$\begin{aligned}\limsup_{r\to\infty}\rho_{\log}^{k}(r) &= \liminf_{r\to\infty}\rho_{\log}^{k}(r)\\ &= \lim_{r\to\infty}\rho_{\log}^{k}(r)\\ &= \rho_{\log}^{k}.\end{aligned}$$

Since

$$\begin{aligned}\log^{k-1}M_f(r) &= (\log r)^{\sigma_{\log}^{k}(r)}\\ &= (\log r)^{\rho_{\log}^{k}(r)}\end{aligned}$$

for a sequence of values of r tending to infinity and

$$\log^{k-1}M_f(r) < (\log r)^{\rho_{\log}^{k}(r)}$$

for remaining r's.

Therefore

$$\limsup_{r\to\infty}\frac{\log^{k-1}M_f(r)}{(\log r)^{\rho_{\log}^{k}(r)}} = 1.$$

Also $\rho_{\log}^{k}(r)$ is differentiable in adjacent intervals.

Also $\left(\rho_{\log}^{k}\right)'(r) = 0$ or $\dfrac{-1}{\prod\limits_{i=0}^{k} log^{i}(r)}$ and then

$$\lim_{r\to\infty}\left(\rho_{\log}^{k}\right)'(r)\prod_{i=0}^{k-1} log^{i}(r) = 0.$$

Finally we obtain $\rho_{\log}^{k}(r)$ is continuous. Hence it proves Case II. □

Example 2.4. If $f(z) = e^{z}$ then its maximum modulus $M_f(r) = \sup\limits_{|z|=r}|f(z)| = e^{r}$.

Define $\phi(r) = \log^{k}M_f(r) > 0$ for sufficiently large values of r.

Clearly $\rho_{\log}^{k} = \limsup\limits_{r\to\infty}\frac{\log\phi(r)}{\log\log r} < \infty$.

Then it can be found (lengthy process) generalised logarithmic proximate order $\rho_{\log}^{k}(r)$ such that

$$\phi(r) \le r^{\rho_{\log}^{k}(r)}$$

for sufficiently large values of r, and

$$\phi(r_n) \ge r^{\rho_{\log}^{k}(r_n)}$$

for a sequence of values of $\{r_n\}$, $r_n \to \infty$.

Corollary 2.5. *If $\alpha > \rho_{\log}^{k}$ then $(r)^{\alpha-\rho_{\log}^{k}(r)}$ is an increasing function of r for all large values of r.*

Proof. For $\rho_{\log}^{k}(r)$ is continuous and the

$$\text{derivative of }(r)^{\alpha-\rho_{\log}^{k}(r)} = (r)^{\alpha-1-\rho_{\log}^{k}(r)}\left[\alpha-\rho_{\log}^{k}(r) + r.\log r.\left(\rho_{\log}^{k}\right)'(r)\right],$$

which will be positive for all large values of r, since $\rho_{\log}^{k}(r)\to\rho_{\log}^{k}$ and $r.\log r.\left(\rho_{\log}^{k}\right)'(r)\to 0$ as $r\to\infty$. □

Theorem 2.6. *For every entire function $f(z)$ of generalised logarithmic order $\rho^k_{\log}$ and generalised logarithmic type $T^k_{\log}$, there exists a generalised logarithmic proximate type $T^k_{\log}(r)$.*

Proof.

$$\begin{aligned}\rho^k_{\log} &= \limsup_{r\to\infty} \frac{\log^k M_f(r)}{\log\log r},\\ T^k_{\log} &= \limsup_{r\to\infty} \frac{\log^{k-1} M_f(r)}{(\log r)^{\rho^k_{\log}}}.\end{aligned}$$

Let

$$S^k_{\log}(r) = \frac{\log^{k-1} M_f(r)}{(\log r)^{\rho^k_{\log}}}.$$

Then there may be two cases arise.

Case I: $S^k_{\log}(r) > T^k_{\log}$ for a sequence of values of r tending to infinity.

Set

$$Q^k_{\log}(r) = \max_{x\geq r_1}\left\{S^k_{\log}(x)\right\}.$$

As $S^k_{\log}(x)$ is continuous, $\limsup\limits_{x\to\infty} S^k_{\log}(x) = T^k_{\log}$ and $S^k_{\log}(x) > T^k_{\log}$ for a sequence of values of x tending to infinity, $Q^k_{\log}(r)$ exists and is a nonincreasing function of r.

Let r_1 be a number such that $r_1 > \exp^k(1)$ and $Q^k_{\log}(r_1) = \max_{x\geq r_1}\left\{S^k_{\log}(x)\right\} = S^k_{\log}(r_1)$. Such values exists for a sequence of values of r tending to infinity.

Next, suppose that $T^k_{\log}(r_1) = Q^k_{\log}(r_1)$ and choose t_1 be the smallest integer not less than $1 + r_1$ such that $Q^k_{\log}(r_1) > Q^k_{\log}(t_1)$.

We define, $T^k_{\log}(r) = T^k_{\log}(r_1) = Q^k_{\log}(r_1)$ for $r_1 < r \leq t_1$.

Set u_1 as

$$\begin{aligned} u_1 &> t_1\\ T^k_{\log}(r) &= T^k_{\log}(r_1) - \log^k r + \log^k t_1 \quad \text{for } t_1 \leq r \leq u_1,\\ T^k_{\log}(r) &= Q^k_{\log}(r) \quad \text{for } r = u_1,\end{aligned}$$

but

$$T^k_{\log}(r) > Q^k_{\log}(r) \text{ for } t_1 \leq r \leq u_1.$$

Let r_2 be the smallest value of r for which $r_2 \geq u_1$ and $Q^k_{\log}(r_2) = S^k_{\log}(r_2)$.

If $r_2 > u_1$ then let $T^k_{\log}(r) = Q^k_{\log}(r)$ for $u_1 \leq r \leq r_2$. One can be easily verified that $T^k_{\log}(r)$ is constant in $u_1 \leq r \leq r_2$.

Repeating the argument we obtain that $T^k_{\log}(r)$ is differentiable in adjacent intervals. Further $\left(T^k_{\log}\right)'(r) = 0$ or $-\left(\prod\limits_{i=0}^{k-1} log^i(r)\right)$ and $T^k_{\log}(r) \geq Q^k_{\log}(r) \geq S^k_{\log}(r)$ for all $r \geq r_1$.

Again $T^k_{\log}(r) = S^k_{\log}(r)$ for an infinite number of values of r, also $T^k_{\log}(r)$ is nonincreasing and $T^k_{\log} = \limsup\limits_{r\to\infty} S^k_{\log}(r) = \lim\limits_{r\to\infty} Q^k_{\log}(r)$.

So,

$$\limsup_{r\to\infty} T^k_{\log}(r) = \liminf_{r\to\infty} T^k_{\log}(r) = \lim_{r\to\infty} T^k_{\log}(r) = T^k_{\log}$$

and

$$\lim_{r\to\infty}\left(T_{\log}^{k}\right)'(r)\prod_{i=0}^{k-2}log^{i}(r)=0.$$

Further we have,

$$M_f(r)=\exp^{k-1}\left\{(\log r)^{\rho_{\log}^{k}}\,S_{\log}^{k}(r)\right\}=\exp^{k-1}\left\{(\log r)^{\rho_{\log}^{k}}\,T_{\log}^{k}(r)\right\}$$

for sufficiently large values of r,

$$M_f(r)<\exp^{k-1}\left\{(\log r)^{\rho_{\log}^{k}}\,T_{\log}^{k}(r)\right\}$$

for the remaining r's.

Therefore

$$\limsup_{r\to\infty}\frac{M_f(r)}{\exp^{k-1}\left\{(\log r)^{\rho_{\log}^{k}}\,T_{\log}^{k}(r)\right\}}=1.$$

Case II: Let $S_{\log}^{k}(r)\leq T_{\log}^{k}$ for sufficiently large values of r. There are two Subcases.

Subcase A:

$$S_{\log}^{k}(r)=T_{\log}^{k}$$

for atleast a sequence of values of r tending to infinity.

We take $T_{\log}^{k}(r)=T_{\log}^{k}$ for all values of r.

Subcase B:

$$S_{\log}^{k}(r)<T_{\log}^{k}$$

for sufficiently large values of r.

Let $L_{\log}^{k}(r)=\max_{X\leq x\leq r}\left\{S_{\log}^{k}(x)\right\}$, where $X>\exp^{k}(1)$ is such that $S_{\log}^{k}(x)<T_{\log}^{k}$ whenever $x\geq X$.

Note that $L_{\log}^{k}(r)$ is nondecreasing. Take a suitably large value of $r_1\geq X$ and let

$$\begin{aligned}T_{\log}^{k}(r_1)&=T_{\log}^{k},\\ T_{\log}^{k}(r)&=T_{\log}^{k}+\log^{k}r-\log^{k}r_1,\ \text{for } s_1\leq r\leq r_1,\end{aligned}$$

where $s_1<r_1$ is such that $L_{\log}^{k}(s_1)=T_{\log}^{k}(s_1)$. If $L_{\log}^{k}(s_1)\neq S_{\log}^{k}(s_1)$ then we take $T_{\log}^{k}(r)=L_{\log}^{k}(r)$ upto the nearest point $t_1<s_1$ at which $L_{\log}^{k}(t_1)=S_{\log}^{k}(t_1)$.

$T_{\log}^{k}(r)$ is then constant for $t_1\leq r\leq s_1$. If $L_{\log}^{k}(s_1)=S_{\log}^{k}(s_1)$, then let $t_1=s_1$.

Choose $r_2>r_1$ suitably large and let

$$\begin{aligned}T_{\log}^{k}(r_2)&=T_{\log}^{k},\\ T_{\log}^{k}(r)&=T_{\log}^{k}+\log^{k}r-\log^{k}r_2,\ \text{for } s_2\leq r\leq r_2\end{aligned}$$

where $s_2\,(<r_2)$ is such that $L_{\log}^{k}(s_2)=T_{\log}^{k}(s_2)$.

If $L_{\log}^{k}(s_2)\neq S_{\log}^{k}(s_2)$ then $L_{\log}^{k}(r)=T_{\log}^{k}(r)$ for $t_2\leq r\leq s_2$ where $t_2\,(<s_2)$ is the nearest point to s_2 at which $L_{\log}^{k}(t_2)=S_{\log}^{k}(t_2)$.

If $L_{\log}^{k}(s_2)=S_{\log}^{k}(s_2)$, then let $t_2=s_2$.

For $r<t_2$, let

$$T_{\log}^{k}(r)=T_{\log}^{k}(t_2)+\log^{k}(t_2)-\log^{k}r\ ,\ \text{for } u_1\leq r\leq t_2$$

where $u_1(<t_2)$ is the point of intersection of $y=T_{\log}^{k}$ with

$$y=T_{\log}^{k}(t_2)+\log^{k}(t_2)-\log^{k}r.$$

Let $T^k_{\log}(r) = T^k_{\log}$ for $r_1 \le r \le u_1$. It is always possible to choose r_2 so large that $r_1 < u_1$.

Repeating the procedure and note that

$$T^k_{\log}(r) \ge L^k_{\log}(r) \ge S^k_{\log}(r)$$

and $T^k_{\log}(r) = S^k_{\log}(r)$ for $r = t_1, t_2, t_3, ...$

Hence

$$\lim_{r\to\infty} T^k_{\log}(r) = T^k_{\log}$$

and

$$\limsup_{r\to\infty} \frac{M_f(r)}{\exp^{k-1}\left\{(\log r)^{\rho^k_{\log}} T^k_{\log}(r)\right\}} = 1.$$

□

It is well known that the estimation of the number of zeros of an entire function of finite order in terms of its order [7]. The same result is true for the poles of a meromorphic function of finite generalised order [3]. Here we prove the following theorem in terms of finite generalised logarithmic order.

Theorem 2.7. *If $f(z)$ be a nonconstant meromorphic function of finite generalised logarithmic order $\rho^k_{\log}$ with $f(0) \ne 0, \infty$ and a be any complex number, finite or infinite. Then for a generalised logarithmic proximate order $\rho^k_{\log}(r)$ of $f(z)$ and for all large r,*

$$\log^{k-2} n(r,a) \le A(r)^{\rho^k_{\log}(r)}$$

where A is a suitable constant independent of a.

Proof. From Nevalinna's first fundamental theorem we get

$$m(r,a) + N(r,a) = T_f(r) + O(1),$$

which implies,

$$N(r,a) \le T_f(r) + O(1).$$

Replacing r by λr $(\lambda > 1)$,

$$N(\lambda r,a) \le T_f(\lambda r) + O(1).$$

Therefore,

$$n(r,a)\log\lambda \le \int_0^{\lambda r} \frac{n(t,a)}{t} dt \le T_f(\lambda r) + O(1).$$

Taking repeated logarithms we get,

$$\log^{k-2} n(r,a) \le \log^{k-2} T_f(\lambda r) + O(1). \tag{2}$$

Also

$$\rho^k_{\log}(r) = \frac{\log^{k-1} T_f(r)}{\log\log r}$$

then for a sequence of values of r tending to infinity we have

$$\log^{k-2} T_f(\exp r) = (r)^{\rho^k_{\log}(r)}.$$

For the remaining r's

$$\log^{k-2} T_f(\exp r) \le (r)^{\rho^k_{\log}(r)}.$$

Therefore we have,

$$\limsup_{r\to\infty} \frac{\log^{k-2} T_f(\exp r)}{(r)^{\rho_{\log}^k(r)}} = 1$$

for given any $\varepsilon > 0$ and for all large values of r we have

$$\log^{k-2} T_f(\exp r) < (1+\varepsilon)(r)^{\rho_{\log}^k(r)}.$$

Then we get,

$$\log^{k-2} T_f(\exp \lambda r) < (1+\varepsilon)(\lambda r)^{\rho_{\log}^k(\lambda r)}. \tag{3}$$

From (2) we have using (3),

$$\begin{aligned}
\log^{k-2} n(r,a) &\le \log^{k-2} T_f(\lambda r) + O(1) \\
&< \log^{k-2} T_f(\exp(\lambda r)) + O(1) \\
&< (1+\varepsilon)(\lambda r)^{\rho_{\log}^k(\lambda r)} + O(1) \\
&= \frac{(1+\varepsilon)(\lambda r)^{\rho_{\log}^k+1}}{(\lambda r)^{\rho_{\log}^k+1-\rho_{\log}^k(\lambda r)}} + O(1).
\end{aligned}$$

Using Corollary 2.5, $(r)^{\rho_{\log}^k+1-\rho_{\log}^k(\lambda r)}$ is increasing for all large r, Then for large r we have from the above relation,

$$\log^{k-2} n(r,a) \le A(r)^{\rho_{\log}^k(r)},$$

where A is defined before. This proves the theorem. □

Also we know that $T_f(r)$ and $\log M_f(r)$ are mutually replaceable in the formula for the order of an entire function $f(z)$. In this section we prove two theorems on comparative growths of $T_f(r)$ and $\log M_f(r)$ in terms of generalised logarithmic proximate order.

THEOREM 2.8. *If $f(z)$ be a nonconstant entire function of finite generalised logarithmic order $\rho_{\log}^k$ and generalised logarithmic proximate order $\rho_{\log}^k(r)$, for $k > 2$*

$$\liminf_{r\to\infty} \frac{\log^{k-1} M_f(\exp r)}{\log^{k-2} T_f(\exp r)} = 1.$$

Proof. We have from (1) putting R by λr,

$$T_f(r) \le \log M_f(r) \le \frac{\lambda+1}{\lambda-1} T_f(\lambda r)$$

Taking repeated logarithms for all large values of r and replacing r by $\exp r$ we have

$$\log^{k-2} T_f(\exp r) \le \log^{k-1} M_f(\exp r) \le \log^{k-2} T_f(\exp \lambda r) + O(1). \tag{4}$$

Using the first part of (4) we have

$$1 \le \liminf_{r\to\infty} \frac{\log^{k-1} M_f(\exp r)}{\log^{k-2} T_f(\exp r)}. \tag{5}$$

From last two inequality of (4) we have,

$$\log^{k-1} M_f(\exp r) \le \log^{k-2} T_f(\exp \lambda r) + O(1). \tag{6}$$

Now by (3) we have,

$$\log^{k-2} T_f(\exp \lambda r) < (1+\varepsilon)(\lambda r)^{\rho^k_{\log}(\lambda r)} \tag{7}$$

Then we have for all large values of r and using (7) in (6),

$$\begin{aligned} \log^{k-1} M_f(\exp r) &< \log^{k-2} T_f(\exp \lambda r) + O(1) \\ &< (1+\varepsilon)(\lambda r)^{\rho^k_{\log}(\lambda r)} + O(1) \\ &= \frac{(1+\varepsilon)(\lambda r)^{\rho^k_{\log}+1}}{(\lambda r)^{\rho^k_{\log}+1-\rho^k_{\log}(\lambda r)}} + O(1). \end{aligned} \tag{8}$$

By Corollary 2.5, $(r)^{\rho^k_{\log}+1-\rho^k_{\log}(\lambda r)}$ is increasing for all large r, then from (8) we get

$$\log^{k-1} M_f(\exp r) \le (1+\varepsilon)\lambda^{\rho^k_{\log}+1}(r)^{\rho^k_{\log}(r)} + O(1). \tag{9}$$

Again since

$$\limsup_{r\to\infty} \frac{\log^{k-2} T_f(\exp r)}{(r)^{\rho^k_{\log}(r)}} = 1,$$

then we get for a sequence of values of r tending to infinity and for arbitrary ε,

$$(1-\varepsilon)(r)^{\rho^k_{\log}(r)} < \log^{k-2} T_f(\exp r).$$

We get from (9) for a sequence of values of r tending to infinity,

$$\log^{k-1} M_f(\exp r) \le \frac{1+\varepsilon}{1-\varepsilon}\lambda^{\rho^k_{\log}+1} \log^{k-2} T_f(\exp r) + O(1).$$

Therefore,

$$\liminf_{r\to\infty} \frac{\log^{k-1} M_f(\exp r)}{\log^{k-2} T_f(\exp r)} \le \frac{1+\varepsilon}{1-\varepsilon}\lambda^{\rho^k_{\log}+1}.$$

Since $\varepsilon\,(0<\varepsilon<1)$ and $\lambda\,(>1)$ is arbitrary, we have

$$\liminf_{r\to\infty} \frac{\log^{k-1} M_f(\exp r)}{\log^{k-2} T_f(\exp r)} \le 1. \tag{10}$$

Combining (5) and (10) we have the theorem. $\square$

Theorem 2.9. *If $P>0$, then for $k\ge 2$*

$$\liminf_{r\to\infty} \frac{\log^{k-1} M_f(\exp r)}{\log^{k-2} T_f(\exp r)\left(\log^{k-1} T_f(\exp r)\right)^P} = 0.$$

Proof. From the above note and Theorem 2.8, we have

$$\liminf_{r\to\infty} \frac{\log^{k-1} M_f(\exp r)}{\log^{k-2} T_f(\exp r)} < \infty.$$

Also

$$\lim_{r\to\infty} \left(\log^{k-1} T_f(\exp r)\right)^P = \infty.$$

Hence the theorem proved. $\square$

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