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ON THE EXTENT OF THE DIVISIBILITY OF FIBONOMIAL COEFFICIENTS BY A PRIME NUMBER

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Abstract. Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence and p be a prime number. For $1 \leq k \leq m$, the Fibonomial coefficient is defined as

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_{m-k+1}...F_{m-1}F_m}{F_1...F_k}$$

and $\begin{bmatrix} m \\ k \end{bmatrix}_F = 0$ when $k > m$. Let a and n be positive integers. In this paper, we find the conditions of prime number p which divides Fibonomial coefficient $\begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F$. Furthermore, we also find the conditions of p when $\begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F$ is not divisible by p.

1. Introduction

Let $(F_n)_{n\geq 0}$ be the Fibonacci sequence given by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0$ and $F_1 = 1$. In 1915, G. Fontené [1] published a note suggesting a generalization of binomial coefficients, replacing natural numbers into an arbitary sequence (A_n) of real or complex numbers. After that there has been much interest in Fibonomial coefficients $\begin{bmatrix} m \\ k \end{bmatrix}_F$ which is defined for $1 \leq k \leq m$ as

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_{m-k+1}...F_{m-1}F_m}{F_1...F_k}$$

and $\begin{bmatrix} m \\ k \end{bmatrix}_F = 0$ when $k > m$. It is shown that Fibonomial coefficient has a integer value which can be proved by the formula

$$\begin{bmatrix} m \\ k \end{bmatrix}_F = F_{k+1}\begin{bmatrix} m-1 \\ k \end{bmatrix}_F + F_{m-k-1}\begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F,$$

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which is a consequence of the formula

$$F_m = F_{k+1}F_{m-k} + F_kF_{m-k-1}.$$

In the recent paper, Diego Marques, James A. Sellers and Pavel Trojovsky [3] proved that if p is a prime number such that $p \equiv 2$ or $-2 \pmod 5$, then $p \mid \begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F$ for all positive integer a and they left a conjecture that if $p \equiv 1$ or $-1 \pmod 5$, then $p \nmid \begin{bmatrix} p^{a+1} \\ p^a \end{bmatrix}_F$ which we shall prove in this paper. Furthermore, we prove the generalization of the conjecture, that is, we find the conditions of prime number p which divides the Fibonomial coefficient $\begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F$, where n is a positive integer. The result is given in the following theorem.

Theorem 1.1. *Let a, n be positive integers and p be a prime number. If $p \equiv 2$ or $-2 \pmod 5$, then*

$$\begin{cases} p \mid \begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F & \text{if } n \equiv 1 \pmod 2, \\ p \nmid \begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F & \text{if } n \equiv 0 \pmod 2, \end{cases}$$

and if $p \equiv 1$ or $-1 \pmod 5$, then

$$p \nmid \begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F .$$

In section 2 and 3, we recall and prove some useful lemmas of the Fibonacci numbers such as a result concerning the p-adic order of F_n and we shall prove the Theorem 1.1 in section 4.

2. Preliminaries

We shall recall some lemmas about the Fibonacci numbers from [3] for the convenience of the readers.

Lemma 2.1. [3, Lemma 2.1] *We have*

1. *$F_n \mid F_m$ if and only if $n \mid m$.*
2. *If $m > k > 1$ then*
$$\begin{bmatrix} m \\ k \end{bmatrix}_F = \frac{F_m}{F_k} \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_F .$$
3. *(d'Ocangne's identity) $(-1)^n F_{m-n} = F_mF_{n+1} - F_nF_{m+1}$.*
4. *For all primes p, $F_{p-(\frac{5}{p})} \equiv 0 \pmod p$, where $(\frac{a}{q})$ denotes the Legendre symbol of a with respect to a prime $q > 2$.*

Before stating the next lemma, we shall define $z(n)$ as the smallest positive integer k such that $n \mid F_k$ for a positive integer n.

Lemma 2.2. [3, Lemma 2.2] *If $n \mid F_m$, then $z(n) \mid m$.*

Let $p \neq 5$ be a prime number. From Lemma 2.1 (4) and Lemma 2.2, we find that $z(p)$ divides $p - \left(\frac{5}{p}\right)$ and it is well-known that $\left(\frac{5}{p}\right) = \pm 1$ according to the residue of p modulo 5. This means $z(p)$ divides $p+1$ or $p-1$.

Lemma 2.3. [3, Lemma 2.3] *For all primes* $p \neq 5$, $gcd(z(p), p) = 1$.

3. The highest power of a prime p

In 1995, Tamás Lengyel [2] has proven the following proposition, but we prove this proposition using another method in this paper.

Proposition 3.1. *For* $n \geq 1$, *we have*

$$\nu_2(F_n) = \begin{cases} 0 & \text{if} \quad n \equiv 1,2 \pmod 3, \\ 1 & \text{if} \quad n \equiv 3 \pmod 6, \\ 3 & \text{if} \quad n \equiv 6 \pmod{12}, \\ \nu_2(n)+2 & \text{if} \quad n \equiv 0 \pmod{12}, \end{cases}$$

$$\nu_5(F_n) = \nu_5(n), \text{and if } p \text{ is a prime number } \neq 2 \text{ or } 5, \text{then}$$

$$\nu_p(F_n) = \begin{cases} \nu_p(n) + \nu_p(F_{z(p)}) & \text{if} \quad n \equiv 0 \pmod{z(p)}, \\ 0 & \text{if} \quad n \not\equiv 0 \pmod{z(p)}. \end{cases}$$

We use the following lemma to prove the proposition 3.1.

Lemma 3.2. *For* $x = a + b\sqrt{n}$, $y = a - b\sqrt{n}$ *and a prime* $p \neq 2$, *we have*

$$\nu_p\left(\frac{x^k - y^k}{\sqrt{n}}\right) = \nu_p(k) + \nu_p\left(\frac{x-y}{\sqrt{n}}\right)$$

for k *is a positive integer and* $p \mid \frac{x-y}{\sqrt{n}}$.

Proof. First, we easily know that $\frac{x^k - y^k}{\sqrt{n}}$ is an integer, since

$$x^k - y^k = (x-y)(x^{k-1} + x^{k-2}y + \cdots + y^{k-1})$$

and $x - y$ is divisible by $\sqrt{n}$. Let us consider the case of $k = p$. Then we have

$$\nu_p\left(\frac{x^p - y^p}{\sqrt{n}}\right) = \nu_p\left(\frac{x-y}{\sqrt{n}}\right) + \nu_p(x^{p-1} + x^{p-2}y + \cdots + y^{p-1}).$$

Since $x^{p-1} + x^{p-2}y + \ldots + y^{p-1} \equiv 0 \pmod p$ and $x^{p-1} + x^{p-2}y + \ldots + y^{p-1} \not\equiv 0 \pmod{p^2}$, we have

$$\nu_p\left(\frac{x^p - y^p}{\sqrt{n}}\right) = \nu_p\left(\frac{x-y}{\sqrt{n}}\right) + 1.$$

Now, let us consider the other case, that is $k = p^\alpha\beta$ with $\gcd(\beta, p) = 1$. In this case, we have

$$\begin{aligned}
\nu_p\left(\frac{x^k - y^k}{\sqrt{n}}\right) &= \nu_p\left(\frac{\left(x^{p^\alpha}\right)^\beta - \left(y^{p^\alpha}\right)^\beta}{\sqrt{n}}\right) = \nu_p\left(\frac{(x^p)^\alpha - (y^p)^\alpha}{\sqrt{n}}\right) \\
&= \nu_p\left(\frac{\left(x^{p^{\alpha-1}}\right)^p - \left(y^{p^{\alpha-1}}\right)^p}{\sqrt{n}}\right) = \nu_p\left(\frac{x^{p^{\alpha-1}} - y^{p^{\alpha-1}}}{\sqrt{n}}\right) + 1 \\
&= \nu_p\left(\frac{x-y}{\sqrt{n}}\right) + \alpha = \nu_p\left(\frac{x-y}{\sqrt{n}}\right) + \nu_p(n).
\end{aligned}$$

Therefore, we have the desired result. □

REMARK 3.3. For $p = 2$, suppose $n = 2^\alpha\beta$ with $\gcd(\beta, 2) = 1$. Proceeding as before, we get

$$\begin{aligned}
\nu_2\left(\frac{x^k - y^k}{\sqrt{n}}\right) &= \nu_2\left(\frac{x^{2^\alpha\beta} - y^{2^\alpha\beta}}{\sqrt{n}}\right) = \nu_2\left(\frac{x^{2^\alpha} - y^{2^\alpha}}{\sqrt{n}}\right) \\
&= \nu_2\left(x^{2^{\alpha-1}} + y^{2^{\alpha-1}}\right) + \nu_2\left(\frac{x^{2^{\alpha-1}} - y^{2^{\alpha-1}}}{\sqrt{n}}\right) \\
&= \nu_2\left(x^{2^{\alpha-1}} + y^{2^{\alpha-1}}\right) + \nu_2\left(x^{2^{\alpha-2}} + y^{2^{\alpha-2}}\right) + \nu_2\left(\frac{x^{2^{\alpha-2}} - y^{2^{\alpha-2}}}{\sqrt{n}}\right).
\end{aligned}$$

This means

$$\nu_2\left(\frac{x^k - y^k}{\sqrt{n}}\right) = \nu_2\left(x^{2^{\alpha-1}} + y^{2^{\alpha-1}}\right) + \nu_2\left(x^{2^{\alpha-2}} + y^{2^{\alpha-2}}\right) + \cdots + \nu_2(x^2 + y^2) + \nu_2(x + y) + \nu_2\left(\frac{x-y}{\sqrt{n}}\right).$$

Now, let us prove the proposition 3.1.

Proof of Proposition 3.1. The following table shows first few elements of the highest power of prime number 2 of F_n modulo 16.

TABLE 1. The highest power of prime number 2 of F_n modulo 16

n	F_n	$\nu_2(F_n)$	n	F_n	$\nu_2(F_n)$	n	F_n	$\nu_2(F_n)$
0	0	0	9	2	1	18	8	3
1	1	0	10	7	0	19	5	0
2	1	0	11	9	0	20	13	0
3	2	1	12	0	0	21	2	1
4	3	0	13	9	0	22	15	0
5	5	0	14	9	0	23	1	0
6	8	3	15	2	1	24	0	0
7	13	0	16	11	0	25	1	0
8	5	0	17	13	0	26	1	0

According to table, we easily find that

$$\nu_2(F_n) = \begin{cases} 0 & \text{if} \quad n \equiv 1, 2 \pmod{3}, \\ 1 & \text{if} \quad n \equiv 3 \pmod{6}, \\ 3 & \text{if} \quad n \equiv 6 \pmod{12}. \end{cases}$$

Now, let us apply the lemma and remark to generalize the power of a prime p of the nth Fibonacci number. According to Binet's formula, the nth Fibonacci number can be expressed as $\frac{\alpha^n-\beta^n}{\sqrt{5}}$, where $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$. Let us compute $\nu_2\left(\frac{(2\alpha)^n-(2\beta)^n}{\sqrt{5}}\right)$ for positive integer n which is divisible by 12. We have the following equation by appying lemma with $p=2$ and $n=2^x y$, where $2 \nmid y$.

$$\begin{aligned}\nu_2\left(\frac{(2\alpha)^n-(2\beta)^n}{\sqrt{5}}\right) &= \nu_2\left(\frac{(2\alpha)^{2^x y}-(2\beta)^{2^x y}}{\sqrt{5}}\right)\\ &= \nu_2\left((2\alpha)^{2^{x-1}y}+(2\beta)^{2^{x-1}y}\right)+\nu_2\left(\frac{(2\alpha)^{2^{x-1}y}-(2\beta)^{2^{x-1}y}}{\sqrt{5}}\right)\end{aligned}$$

and continuing this process, we get

$$\nu_2\left((2\alpha)^{2^{x-1}y}+(2\beta)^{2^{x-1}y}\right)+\cdots+\nu_2\left((2\alpha)^{2y}+(2\beta)^{2y}\right)+\nu_2(2\alpha^y+2\beta^y)+\nu_2\left(\frac{(2\alpha)^y-(2\beta)^y}{\sqrt{5}}\right).$$

This means

$$\nu_2\left(\frac{(2\alpha)^n-(2\beta)^n}{\sqrt{5}}\right) = y2^x+x+2.$$

Therefore,

$$\nu_2\left(\frac{\alpha^n-\beta^n}{\sqrt{5}}\right) = \nu_2\left(\frac{(2\alpha)^n-(2\beta)^n}{\sqrt{5}}\right)-n = (2^x y+x+2)-2^x y = x+2 = \nu_2(n)+2.$$

Next, let us consider the case of $p=5$. $\left(\frac{5}{p}\right)$ is defined only for odd primes except $p=5$, since then the Legendre symbol is not valid. This is the reason why $\nu_5(F_n)$ is different from the other odd primes. We easily find that $z(5)=5$, and $\nu_5(F_n)\geq 1$ if and only if $z(5)=5 \mid n$. Therefore, we have

$$\nu_5\left(\frac{\alpha^n-\beta^n}{\sqrt{5}}\right) = \nu_2\left(\frac{\alpha-\beta}{\sqrt{5}}\right)+\nu_5(n) = 0+\nu_5(n) = \nu_5(n).$$

Lastly, let us consider the case of $p \neq 2,5$. In this case, we also easily know that

$$\nu_p(F_n)\geq 1 \text{ if and only if } z(p)\mid n$$

in a similar way as when $p=5$. Let $A=\alpha^{z(p)}, B=\beta^{z(p)}, n=Nz(p)$. Then we get

$$\nu_p\left(\frac{\alpha^n-\beta^n}{\sqrt{5}}\right) = \nu_p\left(\frac{(\alpha^{z(p)})^N-(\beta^{z(p)})^N}{\sqrt{5}}\right).$$

Now, we can apply the lemma, since $p \mid \frac{\alpha^{z(p)}-\beta^{z(p)}}{\sqrt{5}} = F_{z(p)}$. Hence, we have

$$\begin{aligned}\nu_p(F_N) &= \nu_p\left(\frac{\alpha^{z(p)}-\beta^{z(p)}}{\sqrt{5}}\right)+\nu_p(N) = \nu_p(F_{z(p)})+\nu_p(N)\\ &= \nu_p(F_{z(p)})+\nu_p(n).\end{aligned}$$

Since $z(p)\mid p-1$ or $z(p)\mid p+1$, we obtain $\gcd(z(p),p)=1$ for $p\neq 2,5$. Therefore, $\nu_p(N)=\nu_p(n)$. □

4. Proof of Theorem 1.1

In this section, we prove the Theorem 1.1 which is the generalization of the divisibility of certain Fibonomical coefficients.

Proof of Theorem 1.1. First, let us consider the case of $p \equiv \pm 2 \pmod 5$. We can denote $z(p) = \frac{p+1}{k}$ for some positive integer k, since $z(p) \mid p+1$. We define two sets G_1 and G_2 as

$$\begin{aligned} G_1 &= \{i \mid 1 \le i \le p^a, \;\; z(p) \mid i\}, \\ G_2 &= \{j \mid p^{a+1} - p^a + 1 \le j \le p^{a+n}, \; z(p) \mid j\}. \end{aligned}$$

To prove the Theorem, we only need to compare $\sum_{i \in G_1} \nu_p(F_i)$ and $\sum_{j \in G_2} \nu_p(F_j)$. The proof splits in four cases.

Case 1 : $2 \nmid n$ and $2 \nmid a$

In this case, we have

$$\begin{aligned} G_1 &= \left\{ \frac{p+1}{k}, \frac{2(p+1)}{k}, \cdots, \; p^a + 1 - \frac{p+1}{k} \right\}, \\ G_2 &= \left\{ p^{a+n} - p^a - 2 + \frac{p+1}{k}, \cdots, \; p^{a+n} - 1 \right\}. \end{aligned}$$

Then we obtain

$$\sum_{i \in G_1} \nu_p(F_i) = \sum_{i \in G_1} \nu_p(i) + \sum_{i \in G_1} \nu_p(F_{z(p)})$$

and

$$\sum_{j \in G_2} \nu_p(F_j) = \sum_{j \in G_2} \nu_p(j) + \sum_{j \in G_2} \nu_p(F_{z(p)}).$$

Since $|G_2| = |G_1| + 1$,

$$\sum_{j \in G_2} \nu_p(F_{z(p)}) = \sum_{i \in G_1} \nu_p(F_{z(p)}) + 1.$$

We also observe that G_2 and G_1 are almost the same group when considering the remainder of each number divided by p^a, where G_2 has one more element. Therefore, $\sum_{j \in G_2} \nu_p(j) \ge \sum_{i \in G_1} \nu_p(i) + 1$, and this means $p \mid \begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F$.

Case 2 : $2 \nmid n$ and $2 \mid a$

Let us define G_1 and G_2 similarly as above case. In this case, we have

$$z(p) \mid \left(p^{a+n} + 1 - \frac{p+1}{k} \right).$$

Then we obtain

$$\sum_{j \in G_2} \nu_p(F_{z(p)}) = \sum_{i \in G_1} \nu_p(F_{z(p)}) + 1 \text{ and } \sum_{j \in G_2} \nu_p(j) > \sum_{i \in G_1} \nu_p(i).$$

Therefore, $p \mid \begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F$.

Case 3 : $2 \mid n$ and $2 \nmid a$

In this case, we have

$$\begin{aligned} G_1 &= \left\{ \frac{p+1}{k}, \frac{2(p+1)}{k}, \cdots, \ p^a + 1 - \frac{p+1}{k} \right\}, \\ G_2 &= \left\{ p^{a+n} - p^a + \frac{p+1}{k}, \cdots, \ p^{a+n} + 1 - \frac{p+1}{k} \right\}. \end{aligned}$$

Then we obtain

$$\sum_{i \in G_1} \nu_p(F_i) = \sum_{i \in G_1} \nu_p(F_{z(p)}) + \sum_{i \in G_1} \nu_p(i)$$

and

$$\sum_{j \in G_2} \nu_p(F_j) = \sum_{j \in G_2} \nu_p(F_{z(p)}) + \sum_{j \in G_2} \nu_p(j).$$

Since $|G_1| = |G_2|$,

$$\sum_{j \in G_2} \nu_p(F_{z(p)}) = \sum_{i \in G_1} \nu_p(F_{z(p)}).$$

Now, G_1 and G_2 are the same group comparing the remainder of each number divided by p^a. This means

$$\sum_{j \in G_2} \nu_p(j) = \sum_{i \in G_1} \nu_p(i).$$

Therefore, $\nu_p\left(\begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F\right) = 0.$

Case 4 : $2 \mid n$ and $2 \mid a$

Let us define G_1 and G_2 similarly as above case. In this case, we have

$$z(p) \mid p^{a+n} - 1, \ z(p) \mid p^a - 1.$$

Then we obtain

$$\sum_{j \in G_2} \nu_p(F_{z(p)}) = \sum_{i \in G_1} \nu_p(F_{z(p)})$$

and

$$\sum_{j \in G_2} \nu_p(j) = \sum_{i \in G_1} \nu_p(i).$$

Therefore, $\nu_p\left(\begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F\right) = 0.$

According to above four cases, we have the desired result. Next, we prove the case of $p \equiv \pm 1 \pmod 5$. In this case, we have $z(p) \mid p - 1$. This is the difference from the case of $p \equiv \pm 2 \pmod 5$. Let $z(p) = \frac{p-1}{k}$ for some positive integer k. We define G_1 and G_2 as

$$\begin{aligned} G_1 &= \left\{ \frac{p-1}{k}, \frac{2(p-1)}{k}, \cdots, \ p^a - 1 \right\}, \\ G_2 &= \left\{ p^{a+n} - p^a + \frac{p-1}{k}, \cdots, \ p^{a+n} - 1 \right\}. \end{aligned}$$

We observe that $G_2 = \{i + p^{a+n} - p^a \mid i \in G_1\}$. Since $\nu_p(i + p^{a+n} - p^a) = \nu_p(i)$ for $1 \le i \le p^a - 1$ and $|G_1| = |G_2|$, we have $\nu_p\left(\begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F\right) = 0$. This means $p \nmid \begin{bmatrix} p^{a+n} \\ p^a \end{bmatrix}_F$. □

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