Korean J. Math. **19** (2011), No. 2, pp. 181–189

SUBGROUP ACTIONS AND SOME APPLICATIONS

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Abstract. Let G be a group and X be a nonempty set and H be a subgroup of G. For a given $\phi_G : G \times X \longrightarrow X$, a group action of G on X, we define $\phi_H : H \times X \longrightarrow X$, a subgroup action of H on X, by $\phi_H(h, x) = \phi_G(h, x)$ for all $(h, x) \in H \times X$. In this paper, by considering a subgroup action of H on X, we have some results as follows: (1) If H, K are two normal subgroups of G such that $H \subseteq K \subseteq G$, then for any $x \in X$ $(orb_{\phi_G}(x) : orb_{\phi_H}(x)) = (orb_{\phi_G}(x) : orb_{\phi_K}(x)) = (orb_{\phi_K}(x) : orb_{\phi_H}(x))$; additionally, in case of $K \cap stab_{\phi_G}(x) = \{1\}$, if $(orb_{\phi_G}(x) : orb_{\phi_K}(x))$ and $(orb_{\phi_K}(x) : orb_{\phi_H}(x))$ are both finite, then $(orb_{\phi_G}(x) : orb_{\phi_H}(x))$ is finite; (2) If H is a cyclic subgroup of G and $stab_{\phi_H}(x) \neq \{1\}$ for some $x \in X$, then $orb_{\phi_H}(x)$ is finite.

1. Introduction and basic definitions

The group action is a very useful tool for a classical group theory (in particular, Sylow Theorems) ([5]), Galois theory, ring theory ([1, 2, 3]) and module theory ([6]), etc.

Let G be a group and X be a nonempty set. Let $\phi_G : G \times X \longrightarrow X$ be a group action of G on X. Then for any subgroup H of G, we have a subgroup action of H on X, $\phi_H : H \times X \longrightarrow X$ given by $\phi_H(h, x) = \phi_G(h, x)$ for all $(h, x) \in H \times X$. We define the *orbit* of $x \in X$ under the subgroup action ϕ_H of H on X by $orb_{\phi_H}(x) = \{\phi_H(h, x) : \forall h \in H\}$. We also define the *stabilizer* of x under the subgroup action ϕ_H of H on X by $stab_{\phi_H}(x) = \{h \in H : \phi_H(h, x) = x\}$.

For a given subgroup H of a group G, consider $F = \{\alpha H : \alpha \in G\}$, the collection of all distinct left cosets of H in G. For the convenience

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Received March 15, 2011. Revised May 31, 2011. Accepted June 3, 2011.
2000 Mathematics Subject Classification: 16W22.
Key words and phrases: subgroup action, orbit, stabilizer.
This work was supported by a 2-Year Research Grant of Pusan National University.
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of expression, we denote $\phi_G(\alpha, orb_{\phi_H}(x))$ by $orb_{\phi_{\alpha H}}(x)$. Then we note that $orb_{\phi_G}(x)$ is the union of all $orb_{\phi_{\alpha H}}(x)$ and there exists some subcollection F_1 of F such that $\cup_{\alpha H \in F_1} orb_{\phi_{\alpha H}}(x)$ is a disjoint union of $orb_{\phi_G}(x)$. Denote $|F_1|$ by $(orb_{\phi_G}(x) : orb_{\phi_H}(x))$. Clearly, we note that $|F| = (G : H) \geq (orb_{\phi_G}(x) : orb_{\phi_H}(x))$ where $(G : H)$ is the index of H in G, and if $|orb_{\phi_G}(x)|$ is finite, $|orb_{\phi_H}(x)|$ is finite and $|orb_{\phi_G}(x)| = |orb_{\phi_H}(x)|(orb_{\phi_G}(x) : orb_{\phi_H}(x))$.

Example 1. Let n be a positive integer and $\mathbb{Z}_n$ be the ring of integers of modulo n. Let X_n be the set of all 2×2 nonzero, singular matrices over $\mathbb{Z}_n$, G_n be the general linear group of degree 2 over $\mathbb{Z}_n$ and H_n be the special linear group of degree 2 over $\mathbb{Z}_n$ as a subgroup of G_n, i.e., $\{A \in G_n \mid det(A) = 1\}$. In [4], it was shown that $(G_n : H_n) = \phi(n)$, where $\phi(n)$ is the Euler-ϕ number of n.

Consider a group action of G_n on X_n, $\phi_{G_n} : G_n \times X_n \longrightarrow X_n$ defined by $\phi_{G_n}(g, x) = gx \pmod{n}$ for all $(g, x) \in G_n \times X_n$ and a subgroup action of H_n on X_n, $\phi_{H_n} : H_n \times X_n \longrightarrow X_n$ given by $\phi_{H_n}(h, x) = \phi_{G_n}(h, x)$ for all $(h, x) \in H_n \times X_n$. We compute the followings by a computer programming (using Mathematica Ver. 7):
(1) For $n = 6$;
Note that $G_6 = H_6 \,\dot{\cup}\, \alpha H_6$ with $(G_6 : H_6) = 2$ where $\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \in G_6$. Let $x = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \in X_6$. Then
$orb_{\phi_{H_6}}(x) \,\dot{\cup}\, orb_{\alpha\phi_{H_6}}(x) = orb_{\phi_{G_6}}(x)$ with $|orb_{\phi_{H_6}}(x)| = |orb_{\phi_{\alpha H_6}}(x)| = 72$, and then $|orb_{\phi_{G_6}}(x)| = 144$, and so $(orb_{\phi_{G_6}}(x) : orb_{\phi_{H_6}}(x)) = 2 = (G_6 : H_6)$. On the other hand, $orb_{\phi_{H_6}}(y) = orb_{\phi_{G_6}}(y)$ with $|orb_{\phi_{G_6}}(y)| = 24$, and so $(orb_{\phi_{G_6}}(x) : orb_{\phi_{H_6}}(x)) = 1$.
(2) For $n = 10$;
Note that $(G_{10} : H_{10}) = 4$. Let $x = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, z = \begin{pmatrix} 1 & 1 \\ 0 & 5 \end{pmatrix} \in X_{10}$. Then we compute $|orb_{\phi_{G_{10}}}(x)| = 1,440$, $|orb_{\phi_{H_{10}}}(x)| = 360$, and so $(orb_{\phi_{G_{10}}}(x) : orb_{\phi_{H_{10}}}(x)) = 4 = (G_{10} : H_{10})$; $(orb_{\phi_{G_{10}}}(y) : orb_{\phi_{H_{10}}}(y)) = 1$ with $|orb_{\phi_{G_{10}}}(y)| = |orb_{\phi_{H_{10}}}(y)| = 72$; $(orb_{\phi_{G_{10}}}(z) : orb_{\phi_{H_{10}}}(z)) = 1$ with $|orb_{\phi_{G_{10}}}(z)| = |orb_{\phi_{H_{10}}}(z)| = 144$.

In Section 2, we have shown that for a given a group action ϕ_G of

G on X, (1) if H is a normal subgroup of G such that $(G:H)$ is finite, then $(orb_{\phi_G}(x) : orb_{\phi_H}(x))$ is a divisor of $(G:H)$; (2) if H and K are two normal subgroups of a finite group G such that $H \subseteq K \subseteq G$, then $(orb_{\phi_G}(x) : orb_{\phi_H}(x)) = (orb_{\phi_G}(x) : orb_{\phi_K}(x))(orb_{\phi_K}(x) : orb_{\phi_H}(x))$; in case of $K \cap stab_{\phi_G}(x) = \{1\}$ for some $x \in X$, if $(orb_{\phi_G}(x) : orb_{\phi_K}(x))$ and $(orb_{\phi_K}(x) : orb_{\phi_H}(x))$ are both finite, then $(orb_{\phi_G}(x) : orb_{\phi_H}(x))$ is finite; moreover, $(orb_{\phi_G}(x) : orb_{\phi_K}(x))(orb_{\phi_K}(x) : orb_{\phi_H}(x)) = (orb_{\phi_G}(x) : orb_{\phi_H}(x))$.

Let R be a ring with identity, X be the of all nonzero, nonunits of R and G be the group of all units of R. In Section 3, by applying the result obtained in section 2 to the subgroup action $\phi_H : H \times X \longrightarrow X$ for a given subgroup H of G we have shown that if H is a cyclic subgroup of G and $stab_{\phi_H}(x) \neq \{1\}$ for some $x \in X$, then $orb_{\phi_H}(x)$ is finite; if H is infinite, then the converse holds.

2. Subgroup action

We denote the cardinality of a set S by $|S|$. Also write $A \cdot B = \{ab | a \in A, b \in B\}$ for any sets A, B.

Lemma 2.1. *Let ϕ_G be a group action of a group G on a set X. Then $|orb_{\phi_H}(x)| = |orb_{\phi_{\alpha H}}(x)|$ for all cosets αH of H in G.*

Proof. Define $f : orb_{\phi_H}(x) \longrightarrow orb_{\phi_{\alpha H}}(x)$ by $f(\phi_H(h,x)) = \phi_G(\alpha h, x)$ for all $(h,x) \in orb_{\phi_H}(x)$. Then clearly f is well-defined and onto. To show that f is one to one, let $f(\phi_G(h,x)) = f(\phi_G(h_1,x))$ for some $h, h_1 \in H$, and so $\phi_G(\alpha h, x) = \phi_G(\alpha h_1, x)$. Then

$$\begin{aligned}
\phi_H(h,x) &= \phi_G(1, \phi_H(h,x)) = \phi_G(\alpha^{-1}\alpha, \phi_H(h,x)) \\
&= \phi_G(\alpha^{-1}, \phi_G(\alpha, \phi_H(h,x)) \\
&= \phi_G(\alpha^{-1}, \phi_G(\alpha, \phi_H(h_1,x)) \\
&= \phi_G(\alpha^{-1}\alpha, \phi_H(h_1,x)) \\
&= \phi_G(1, \phi_H(h_1,x)) = \phi_H(h_1,x)
\end{aligned}$$

and thus f is one to one. Therefore, f is bijective and so we have the result.

□

Corollary 2.2. *Let ϕ_G be a group action of a group G on a set X and H be a normal subgroup of G. Then $(orb_{\phi_G}(x) : orb_{\phi_H}(x)) = (G : H \cdot stab_{\phi_G}(x))$ for all $x \in X$.*

Proof. By Lemma 2.1, $orb_{\phi_{\alpha H}}(x) = orb_{\phi_{\beta H}}(x)$ for some cosets $\alpha H, \beta H$ of H in G if and only if $\alpha^{-1}\beta \in H \cdot stab_{\phi_G}(x)$. Since H is a normal subgroup of G, $H \cdot stab_{\phi_G}(x)$ is a subgroup of G, and so $(orb_{\phi_G}(x) : orb_{\phi_H}(x)) = (G : H \cdot stab_{\phi_G}(x))$. □

Remark 1. *Let ϕ_G be a group action of a group G on a set X and H be a normal subgroup of G. By Corollary 2.2, we note that for all $x \in X$, (1) $(orb_{\phi_G}(x) : orb_{\phi_H}(x)) = (G : H)$ if and only if $stab_{\phi_G}(x) \subseteq H$; (2) if $(G : H)$ is finite, then $(orb_{\phi_G}(x) : orb_{\phi_H}(x))$ is a divisor of $(G : H)$.*

Corollary 2.3. *Let ϕ_G be a group action of a group G on a set X. Then for all $x \in X$, $|orb_{\phi_G}(x)| = (G : stab_{\phi_G}(x))$.*

Proof. Let $H = \{1\}$. Then it follows from Corollary 2.2. □

Theorem 2.4. *Let ϕ_G be a group action of a group G on a set X and H, K be two subgroups of G. Then (1) $orb_{\phi_H}(x) = orb_{\phi_K}(x)$ for some $x \in X$ if and only if $H \subseteq K \cdot stab_{\phi_G}(x)$ and $K \subseteq H \cdot stab_{\phi_G}(x)$; (2) in particular, if $stab_{\phi_G}(x) = \{1\}$ for some $x \in X$, then $orb_{\phi_H}(x) = orb_{\phi_K}(x)$ if and only if $H = K$.*

Proof. (1). Suppose that $orb_{\phi_H}(x) = orb_{\phi_K}(x)$. Let $h \in H$ be arbitrary. Since $\phi_H(h, x) \in orb_{\phi_H}(x) = orb_{\phi_K}(x)$, $\phi_H(h, x) = \phi_K(k, x)$ for some $k \in K$. Thus $k^{-1}h \in stab_{\phi_G}(x)$, and so $h \in K \cdot stab_{\phi_G}(x)$. Hence $H \subseteq K \cdot stab_{\phi_G}(x)$. Similarly, we have $K \subseteq H \cdot stab_{\phi_G}(x)$.

Conversely, suppose that $H \subseteq K \cdot stab_{\phi_G}(x)$ and $K \subseteq H \cdot stab_{\phi_G}(x)$. Let $\phi_H(h, x) \in orb_{\phi_H}(x)$ be arbitrary. Then $h = kg$ for some $k \in K$ and some $g \in stab_{\phi_G}(x)$. Thus $\phi_H(h, x) = \phi_G(h, x) = \phi_G(kg, x) = \phi_G(k, \phi_G(g, x)) = \phi_G(k, x) = \phi_K(k, x) \in orb_{\phi_K}(x)$, and so $orb_{\phi_H}(x) \subseteq orb_{\phi_K}(x)$. Similarly, we have $orb_{\phi_K}(x) \subseteq orb_{\phi_H}(x)$.

(2). In particular, if $stab_{\phi_G}(x) = \{1\}$, then $orb_{\phi_H}(x) = orb_{\phi_K}(x)$ if and only if $H = K$ by (1). □

Remark 2. *Let ϕ_G be a group action of a group G on a set X and H, K be two subgroups of G. By Theorem 2.4, we note that for some $x \in X$, (1) $orb_{\phi_H}(x) = orb_{\phi_G}(x)$ if and only if $G = H \cdot stab_{\phi_G}(x)$; (2) if $stab_{\phi_G}(x) \subseteq H \cap K$ for some $x \in X$, then $orb_{\phi_H}(x) \cap orb_{\phi_K}(x) = orb_{\phi_{H\cap K}}(x)$. Indeed, clearly, $orb_{\phi_{H\cap K}}(x) \subseteq orb_{\phi_H}(x) \cap orb_{\phi_K}(x)$. Let $y \in orb_{\phi_H}(x) \cap orb_{\phi_K}(x)$ be arbitrary. Then $y = \phi_G(h, x) = \phi_H(h, x) = \phi_K(k, x) = \phi_G(k, x)$ for some $h \in H, k \in K$. Thus $\phi_G(k^{-1}h, x) = x$, and so $k^{-1}h \in stab_{\phi_G}(x)) \subseteq H \cap K$. Hence $h = k(k^{-1}h) \in K(H \cap K) \subseteq KK = K$, and so $y = \phi_G(h, x) \in orb_{\phi_{H\cap K}}(x)$, and then $orb_{\phi_H}(x) \cap orb_{\phi_K}(x) \subseteq orb_{\phi_{H\cap K}}(x)$. Therefore, $orb_{\phi_H}(x) \cap orb_{\phi_K}(x) = orb_{\phi_{H\cap K}}(x)$.*

Theorem 2.5. *Let ϕ_G be a group action of a finite group G on a set X and H, K be two normal subgroups of G such that $H \subseteq K \subseteq G$. Then $(orb_{\phi_G}(x) : orb_{\phi_H}(x)) = (orb_{\phi_G}(x) : orb_{\phi_K}(x))(orb_{\phi_K}(x) : orb_{\phi_H}(x))$.*

Proof. Since G is finite, both $(orb_{\phi_G}(x) : orb_{\phi_K}(x))$ and $(orb_{\phi_K}(x) : orb_{\phi_H}(x))$ are finite. By Corollary 2.2, we have $(orb_{\phi_G}(x) : orb_{\phi_H}(x)) = (G : H \cdot stab_{\phi_G}(x)), (orb_{\phi_K}(x) : orb_{\phi_H}(x)) = (K : H \cdot stab_{\phi_K}(x))$ and $(orb_{\phi_G}(x) : orb_{\phi_K}(x)) = (G : K \cdot stab_{\phi_G}(x))$. We will show that $(K : H \cdot stab_{\phi_K}(x)) = (K \cdot stab_{\phi_G}(x) : K \cdot stab_{\phi_G}(x) : H \cdot stab_{\phi_G}(x))$. Indeed,

$$
\begin{aligned}
(K \cdot stab_{\phi_G}(x) : H \cdot stab_{\phi_G}(x)) &= \frac{|K \cdot stab_{\phi_G}(x)|}{|H \cdot stab_{\phi_G}(x)|} \\
&= (\frac{|K|}{|H|})(\frac{|H \cap stab_{\phi_G}(x)|}{|K \cap stab_{\phi_G}(x)|}) \\
&= (\frac{|K|}{|H|})(\frac{|stab_{\phi_H}(x)|}{|stab_{\phi_K}(x)|})
\end{aligned}
$$

On the other hand,

$$\begin{aligned}(K : H \cdot stab_{\phi_K}(x)) &= \frac{|K|}{|H \cdot stab_{\phi_K}(x)|}\\ &= (\frac{|K|}{|H|})(\frac{|H \cap stab_{\phi_K}(x)|}{|stab_{\phi_K}(x)|})\\ &= (\frac{|K|}{|H|})(\frac{|stab_{\phi_H}(x)|}{|stab_{\phi_K}(x)|})\end{aligned}$$

Hence we have $(K : H\cdot stab_{\phi_K}(x)) = (K\cdot stab_{\phi_G}(x) : H\cdot stab_{\phi_G}(x))$. Therefore, $(orb_{\phi_G}(x) : orb_{\phi_H}(x)) = (G : H \cdot stab_{\phi_G}(x)) = (G : K \cdot stab_{\phi_G}(x))(K \cdot stab_{\phi_G}(x) : H \cdot stab_{\phi_G}(x)) = (G : K \cdot stab_{\phi_G}(x))(K : H \cdot stab_{\phi_K}(x)) = (orb_{\phi_G}(x) : orb_{\phi_K}(x))(orb_{\phi_K}(x) : orb_{\phi_H}(x))$.

□

Lemma 2.6. *Let H, K be normal subgroups of a group G such that $H \subseteq K$. If $(K : H)$ is finite and $K \cap L = \{1\}$ for some subgroup L of G, then $(K : H) = (KL : HL)$.*

Proof. Let $\{k_iH : i = 1, \cdots, r\}$ be the collection of distinct cosets of H in K. Let $k\ell \in KL(k \in K, \ell \in L)$ be arbitrary. Then $k \in k_iH$ for some $k_i \in K$, and so $k\ell \in k_iHL$. Thus $KL = k_1HL \cup \cdots \cup k_rHL$. We will show that $\{k_iHL : i = 1, \cdots, r\}$ is the collection of distinct cosets of HL in KL. Assume that $k_iHL \cap k_jHL \neq \emptyset$ for some $k_i, k_j \in K(k_i \neq k_j)$. Let $a \in k_iHL \cap k_jHL$. Then $a = k_ih_1\ell_1 = k_ih_2\ell_2$ for some $h_1, h_2 \in H, \ell_1, \ell_2 \in L$, and so $(k_ih_1)^{-1}(k_jh_2) = \ell_1\ell_2^{-1} \in (KH) \cap L = K \cap L$. Since $K \cap L = \{1\}$, $k_ih_1 = k_jh_2 \in k_iH \cap k_jH$, a contradiction. Hence $\{k_iHL : i = 1, \cdots, r\}$ is also the collection of distinct cosets of HL in KL, and so $(K : H) = (KL : HL)$. □

Theorem 2.7. *Let ϕ_G be a group action of a group G on a set X and H, K be two normal subgroups of G such that $H \subseteq K \subseteq G$ and $K \cap stab_{\phi_G}(x) = \{1\}$ for some $x \in X$. If both $(orb_{\phi_G}(x) : orb_{\phi_K}(x))$ and $(orb_{\phi_K}(x) : orb_{\phi_H}(x))$ are finite, then $(orb_{\phi_G}(x) : orb_{\phi_H}(x))$ is finite. Moreover, $(orb_{\phi_G}(x) : orb_{\phi_K}(x))(orb_{\phi_K}(x) : orb_{\phi_H}(x)) = (orb_{\phi_G}(x) : orb_{\phi_H}(x))$.*

Proof. By Corollary 2.2, we have $(orb_{\phi_K}(x) : orb_{\phi_H}(x)) = (G : H \cdot stab_{\phi_G}(x))$,
$(orb_{\phi_K}(x) : orb_{\phi_H}(x)) = (K : H{\cdot}stab_{\phi_K}(x))$ and $(orb_{\phi_G}(x) : orb_{\phi_K}(x)) = (G : K \cdot stab_{\phi_G}(x))$. Since H, K are normal subgroups of G such that $H \subseteq K \subseteq G$ and $K \cap stab_{\phi_G}(x) = \{1\}$, $(K : H \cdot stab_{\phi_K}(x)) = (K{\cdot}stab_{\phi_G}(x) : H{\cdot}stab_{\phi_G}(x))$ by Lemma 2.6. Therefore, as in the proof of Theorem 2.5 we have $(orb_{\phi_G}(x) : orb_{\phi_K}(x))(orb_{\phi_K}(x) : orb_{\phi_H}(x)) = (orb_{\phi_G}(x) : orb_{\phi_H}(x))$. □

3. Cyclic subgroup action and some applications

THEOREM 3.1. *Let H be a cyclic subgroup of a group G and ϕ_H be a subgroup action of H on X. If $stab_{\phi_H}(x) \neq \{1\}$ for some $x \in X$, then $orb_{\phi_H}(x)$ is finite.*

Proof. Let $H =< a >$ be a cyclic group generated by $a \in G$. If $orb_{\phi_H}(x) = \{x\}$ or $H = \{1\}$, then $|orb_{\phi_H}(x)| = 1$, and so $orb_{\phi_H}(x)$ is finite. Thus suppose that $orb_{\phi_H}(x) \neq \{x\}$ and $H \neq \{1\}$. Then $|orb_{\phi_H}(x)| \geq 2$. Let $H_0 = stab_{\phi_H}(x)$. Then $H > H_0 \neq \{1\}$, and so $H_0 =< a^t >$ is a proper subgroup of H generated by a^t for some positive integer $t \geq 2$. Let $\phi_H(h, x) \in orb_{\phi_H}(x)$ be arbitrary. Then $h = a^s$ for some integer s. By the division algorithm on $\mathbb{Z}$, the ring of integers, $s = qt + r$ for some $q, r \in \mathbb{Z}$ where $0 \leq r \leq t - 1$. Since $a^t \in H_0 = stab_{\phi_H}(x)$, $x = \phi_H(a^t, x)$, and so $\phi_H(a^{2t}, x) = \phi_H(a^t, \phi_H(a^t, x)) = \phi_H(a^t, x) = x$. Thus by continuing in this process inductively, we have $x = \phi_H(a^t, x) = \cdots = \phi_H(a^{qt}, x)$. Hence $\phi_H(h, x) = \phi_H(a^s, x) = \phi_H(a^{qt+r}, x) = \phi_H(a^r, \phi_H(a^{qt}, x)) = \phi_H(a^r, x)$, and so $orb_{\phi_H}(x) = \{x, \phi_H(a, x), \cdots, \phi_H(a^{t-1}, x)\}$ is finite. □

COROLLARY 3.2. *Let H be an infinite cyclic subgroup of a group G and ϕ_H be a subgroup action of H on X. Then $stab_{\phi_H}(x) \neq \{1\}$ for some $x \in X$ if and only if $orb_{\phi_H}(x)$ is finite.*

Proof. If follows from Corollary 2.3 and Theorem 3.1. □

In this section, let R be a ring with identity, $X(R)$ (simply denoted by X) be the set of all nonzero, nonunits of R and $G(R)$ (simply denoted by G) be the group of all units of R. Let H be a subgroup of G. Then the map $\phi^r_H : H \times X \longrightarrow X$ (resp. $\phi^c_H : H \times X \longrightarrow X$) defined by $\phi^r_H((h,x) = hx$ (resp. $\phi^c_H((h,x) = hxh^{-1})$ is a subgroup action of H on X, which is called the regular action (resp. conjugate action)(refer [1], [2] and [3]). By Theorem 3.1, if H is a cyclic subgroup of G and $stab_{\phi^r_H}(x) \neq \{1\}$ (resp. $stab_{\phi^c_H}(x) \neq \{1\}$) for some $x \in X$, then $orb_{\phi^r_H}(x)$ (resp. $orb_{\phi^c_H}(x)$) is finite.

Recall that the *index* of a nilpotent $x \in R$ is the least positive integer n such that $x^n = 0 \neq x^{n-1}$ and is denoted by $\mathrm{ind}(x)$.

Corollary 3.3. *Let R be a ring and $x \in X$ be a nilpotent with ind(x) $= n$. Then $orb_{\phi^r_H}(x)$ (resp. $orb_{\phi^c_H}(x)$) is finite where H is a cyclic subgroup of G generated by $1 + x^{n-1}$. In particular, if G is cyclic, then $orb_{\phi^r_G}(x)$ (resp. $orb_{\phi^c_G}(x)$) is finite.*

Proof. Since $x \in X$ is nilpotent with $\mathrm{ind}(x) = n$, $1 \neq 1 + x^{n-1} \in H$ and so $(1+x^{n-1})x = x$ (resp. $(1+x^{n-1})x = x(1+x^{n-1})$), which implies that $1 + x^{n-1} \in stab_{\phi^r_H}(x) \neq \{1\}$ (resp. $1 + x^{n-1} \in stab_{\phi^c_H}(x) \neq \{1\}$). Thus $orb_{\phi^r_G}(x)$ (resp. $orb_{\phi^c_G}(x)$) is finite by Theorem 3.1. In particular, if G is cyclic, then $orb_{\phi^r_G}(x)$ (resp. $orb_{\phi^c_G}(x)$) is finite by the similar argument.

□

Corollary 3.4. *Let R be a ring such that $2 \in G$ and $e \in X$ be an idempotent. Then $orb_{\phi^r_H}(e)$ (resp. $orb_{\phi^c_H}(e)$) is finite where H is a cyclic subgroup of G generated by $2e - 1$. In particular, if G is cyclic, then $orb_{\phi^r_G}(e)$ (resp. $orb_{\phi^c_G}(e)$) is finite.*

Proof. Since $2 \in G$, $2e-1 \in G$ and $(2e-1)e = e$ (resp. $(2e-1)e = e(2e-1)$), and so $stab_{\phi^r_H}(e) \neq \{1\}$ (resp. $stab_{\phi^c_H}(e) \neq \{1\}$). Thus $orb_{\phi^r_H}(e)$ (resp. $orb_{\phi^c_H}(e)$) is finite by Theorem 3.1. In particular, if G is cyclic, then $orb_{\phi^r_G}(e)$ (resp. $orb_{\phi^c_G}(e)$) is finite by the similar argument. □

Corollary 3.5. *Let R be a ring and H be a cyclic normal subgroup of G. If $(G : H)$ is finite and $stab_{\phi^r_H}(x) \neq \{1\}$ for some $x \in X$, then $orb_{\phi^r_G}(x)$ is finite.*

Proof. If follows from Corollary 2.3 and Theorem 3.1. □

Acknowledgements. The authors would like to thank the referee for his/her careful checking of the details and helpful comments for making the paper more readable.

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