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ON SENDOV'S CONJECTURE ABOUT CRITICAL POINTS OF A POLYNOMIAL

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Abstract. The derivative of a polynomial $p(z)$ of degree n, with respect to point α is defined by $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$. Let $p(z)$ be a polynomial having all its zeros in the unit disk $|z| \leq 1$. The Sendov conjecture asserts that if all the zeros of a polynomial $p(z)$ lie in the closed unit disk, then there must be a zero of $p'(z)$ within unit distance of each zero. In this paper, we obtain certain results concerning the location of the zeros of $D_\alpha p(z)$ with respect to a specific zero of $p(z)$ and a stronger result than Sendov conjecture is obtained. Further, a result is obtained for zeros of higher derivatives of polynomials having multiple roots.

1. Introduction

The Gauss-Lucas Theorem states that if $\mathbb{D}$ is the set of zeros of a polynomial

$$f(z) = \prod_{\nu=1}^{n}(z - z_\nu)$$

then every zero of the derivative $f'(z)$ is contained in the convex hull containing $\mathbb{D}$. It should be noted that, the set $\mathbb{D}$ cannot be squeezed further in the sense that, no proper subset of $\mathbb{D}$ can be guarranted to contain even one zero of $f'(z)$. Given any proper subset of $\mathbb{D}$, one can easily construct a polynomial whose zeros and critical points lie outside it, as can be seen in case $f(z) = (z-1)^n$. Gauss-Lucas Theorem considers relative position of all the zeros and the critical points of $f(z)$. However, there is a related question that deserves attention, namely given one specific zero z_0 of $f(z)$, what can be said about a neighbourhood of z_0 that will always contain a zero of $f'(z)$. A possible answer is given by famous conjecture known in literature as "Sendov Conjecture."

Sendov Conjecture: If $p(z)$ is a n^{th} degree polynomial having all its zeros in the unit disk $|z| \leq 1$ and if a is any one such zero, then the disk $|z - a| \leq 1$ contains at least one critical point of $f(z)$.

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This conjecture, in general is still an open problem. However, this has been observed that the conjecture is true for some special class of polyunomials. In full generality, the conjecture has been proved only for the polynomials of degree ≤ 8 [3]. Also, the conjecture is true for class of polynomials having a zero at origin.
Let us recall the famous Biernack's Theorem [5, p. 227] which asserts that, if in the situation of Sendov's conjecture, one of the closed unit discs centered at any one of the n zeros contains all the critical points of $f(z)$, then each of the remaining such $n-1$ closed unit discs contains at least one critical point of $f(z)$. This substantiates the fact that the Sendov's conjecture is true whenever polynomial vanishes at the origin.
In [2] Bojanov et al proved an interesting result on verifying asymptotic behavior of degree and zeros of a polynomial in connection to Sendov conjecture by proving the following result.

THEOREM 1.1. *If* $f(z)=\prod_{\nu=1}^{n}(z-z_\nu)$ *has all its zeros in the disc* $|z|\leq 1$, *then each of the discs centered at* z_k *with radius* $(1+|z_1z_2...z_n|)^{\frac{1}{n}}$, $k=1,2,...,n$ *contains at least one zero of* $f'(z)$.

REMARK 1.2. If $f(z)$ has all its zeros in the disc $|z|\leq 1$ and $f(0)=0$, then the product $\prod_{\nu=1}^{n} z_\nu$ of all the zeros, is equal to zero. Hence by above result, the disc $|z-z_\nu|\leq 1$ contains at least one zero of $f'(z)$ for all $\nu=1,2,...,n$. That again validates the Sendov conjecture for polynomials having a zero at the origin.

Thus from Biernacki Theorem [5, p. 227] and Remark 1.2, we conclude that Sendov Conjecture holds in general for the class of polynomials having a zero at the origin. Furthermore, we note that the transformation $z\rightarrow e^{i\theta}z$, does not effect the relative configuration of the zeros and critical points of a polynomial $p(z)$. Hence we may set $z_k=a$, where $0\leq a\leq 1$. With this normalization the Sendov conjecture can now be reformulated as
" Let $p(z)=(z-a)\prod_{j=1}^{n-1}(z-z_j)$, $0<a\leq 1$ and $|z_j|\leq 1$, $1\leq j\leq n-1$, then the disk $|z-a|\leq 1$ contains a critical point of $p(z)$."
Recently, conjecture is proved for sufficiently high degree polynomials by Terence Tao [7] by proving

THEOREM 1.3. *Sendov's conjecture is true for all sufficiently large* n. *That is, there exists an absolute constant* n_0 *such that Sendov's conjecture holds for* $n\geq n_0$.

DEFINITION 1.4. Let $p(z)$ be a polynomial of degree n with complex coefficients and $\alpha\in\mathbb{C}$ be a complex number, then the polynomial

$$D_\alpha p(z)=np(z)+(\alpha-z)p'(z)$$

is called polar derivative of $p(z)$ with pole α. It is a polynomial of degree $n-1$.

Also,

$$lim_{\alpha\rightarrow\infty}\left[\frac{1}{\alpha}D_\alpha p(z)\right]=p'(z)$$

.

Thus, $p'(z)$ can be regarded as the derivative of $p(z)$ with respect to $\alpha=\infty$. Hence theorems about $D_\alpha p(z)$ can be regarded as generalisation for theorems about the derivative of $p(z)$.
Laguerre [5, p. 98] proved the following result for polar derivatives.

THEOREM 1.5. (LAGUERRE) *Let $p(z)$ be a polynomial of degree $n \geq 2$, and let $\alpha \in \mathbb{C}$.*

1. *A circular domain containing all the zeros of $p(z)$, but not the point α, contains all the zeros of the polar derivative $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$.*
2. *let $w \neq \alpha$ be a zero of $D_\alpha p(z)$ such that $p(w) \neq 0$. Then every circle C passing through α and w separates at least two zeros of $p(z)$ unless all the zeros lie on C.*

2. Main Results

Our first result determines the disk containing a zero of $D_\alpha p(z)$ with respect to a specific zero of $p(z)$.

THEOREM 2.1. *Let $p(z) = (z - a)\prod_{k=1}^{n-1}(z - z_k)$, with $|z_k| \leq a$, for all k, $1 \leq k \leq n-1$, $0 < a \leq 1$. Then the polar derivative $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$, $\alpha > 1$ has at least one zero in the disk*

$$\left| z - \frac{a}{2} - \frac{a^2}{2\alpha} \right| \leq \frac{a}{2} - \frac{a^2}{2\alpha} \tag{1}$$

Proof. We have

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$$

and

$$p(z) = (z - a)Q(z)$$

where $Q(z) = \prod_{k=1}^{n-1}(z - z_k)$, then

$$\frac{D_\alpha p(z)'}{D_\alpha p(z)} = \frac{(n-1)p'(z) + (\alpha - z)p''(z)}{np(z) + (\alpha - z)p'(z)}$$

If $z = a$ is a multiple zero of $p(z)$, then $z = a$ is also a zero of $p'(z)$ and therefore $z = a$ is also a zero of $D_\alpha p(z)$. Since $z = a$ lies in (1), therefore the assertion is true in this case. Hence we assume $z = a$ is a simple zero of $p(z)$.
Now, we have.

$$\frac{D_\alpha p(a)'}{D_\alpha p(a)} = \frac{(n-1)p'(a) + (\alpha - a)p''(a)}{np(a) + (\alpha - a)p'(a)}$$

This gives

$$\frac{D_\alpha p(a)'}{D_\alpha p(a)} = \frac{n-1}{\alpha - a} + \frac{2Q'(a)}{Q(a)} \tag{2}$$

If $w_1, w_2, ..., w_{n-1}$ are the zeros of $D_\alpha p(z)$ and $z_1, z_2, ..., z_{n-1}$ are the zeros of $Q(z)$, then from (2), we obtain

$$\sum_{j=1}^{n-1} \frac{1}{a - w_j} = \frac{n-1}{\alpha - a} + 2\sum_{j=1}^{n-1} \frac{1}{a - z_j} \tag{3}$$

Since $|z_j| \le a$, $1 \le j \le n-1$, we have

$$\Re\frac{1}{a-z_j} \ge \frac{1}{2a} \quad \forall \quad 1 \le j \le n-1$$

Taking real parts on both sides of (3), we get

$$\sum_{j=1}^{n-1} \Re\frac{1}{a-w_j} = \frac{n-1}{\alpha-a} + 2\sum_{j=1}^{n-1} \Re\frac{1}{a-z_j}$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{n-1} \Re\frac{1}{a-w_j} &\ge \frac{n-1}{\alpha-a} + 2\frac{n-1}{2a}\\ &= \frac{n-1}{\alpha-a} + \frac{n-1}{a}\\ &= (n-1)\left(\frac{1}{\alpha-a} + \frac{1}{a}\right)\\ \implies \sum_{j=1}^{n-1} \Re\frac{1}{a-w_j} &\ge (n-1)\left(\frac{\alpha}{a(\alpha-a)}\right).\end{aligned}$$

Let $\Re\frac{1}{a-w} = \max_{1\le j\le n-1} \Re\frac{1}{a-w_j}$, then

$$\begin{aligned}\Re\frac{1}{a-w} &\ge \frac{1}{n-1}\sum_{j=1}^{n-1} \Re\frac{1}{a-w_j}\\ &\ge \frac{\alpha}{a(\alpha-a)}.\end{aligned}$$

So,

$$\Re\frac{1}{a-w} \ge \frac{\alpha}{a(\alpha-a)}.$$

implies

$$\left|w - \frac{a}{2} - \frac{a^2}{2\alpha}\right| \le \frac{a}{2} - \frac{a^2}{2\alpha}.$$

This completes the proof of theorem. □

For $a = 1$ and $\alpha = \infty$, we get the following result of Goodman, Rahman and Ratti [4].

Corollary 2.2. *If all the zeros of the polynomial $p(z)$ lie in $|z| \le 1$ and $p(1) = 0$, then $p'(z)$ has at least one zero in*

$$\left|z - \frac{1}{2}\right| \le \frac{1}{2}.$$

For $a = 1$, we get the following result of A. Aziz [1].

Corollary 2.3. *If all the zeros of the polynomial $p(z)$ lie in $|z| \leq 1$ and $p(1) = 0$, then the polynomial $D_\alpha p(z) = np(z) + (\alpha - z)p'(z)$ has at least one zero in*

$$\left| z - \frac{1}{2} - \frac{1}{2\alpha} \right| \leq \frac{1}{2} - \frac{1}{2\alpha}.$$

Using theorem 2.1, we obtain the following result.

Theorem 2.4. *Let $p(z) = (z - a)\prod_{j=1}^{n-1}(z - z_j)$, $0 < a \leq 1$ and $|z_j| \leq a$, $1 \leq j \leq n-1$, then the disk $\left|z - \frac{a}{2}\right| \leq \frac{a}{2}$ contains a critical point of $p(z)$.*

Proof. From the Theorem 2.1, we have shown that the disk

$$\left| z - \frac{a}{2} - \frac{a^2}{2\alpha} \right| \leq \frac{a}{2} - \frac{a^2}{2\alpha}, \quad where \quad \alpha > 1$$

contains at least one zero of $D_\alpha p(z)$. Letting $\alpha \to \infty$ and noting that

$$lim_{\alpha \to \infty} \left[\frac{1}{\alpha} D_\alpha p(z) \right] = p'(z).$$

we conclude that the disk

$$\left| z - \frac{a}{2} \right| \leq \frac{a}{2}$$

contains at least one zero of $p'(z)$.

□

Remark 2.5. Since the disk $\left|z - \frac{a}{2}\right| \leq \frac{a}{2}$ is contained in the disk $|z - a| \leq 1$. Hence $|z - a| \leq 1$ contains at least one zero of $p'(z)$. Thus, we get a smaller region which contains a zero of $p'(z)$ than the region predicted by Sendov.

Remark 2.6. In the statement of Theorem 2.4, we can start with any other zero z_j instead of "a" and then apply the transformation $z \to e^{-i\theta}z$, where $\theta = arg z_j$, so that $z_j \to a$, where $a \in (0, 1]$ and note that configuration of zeros and critical points is preserved under this transformation.

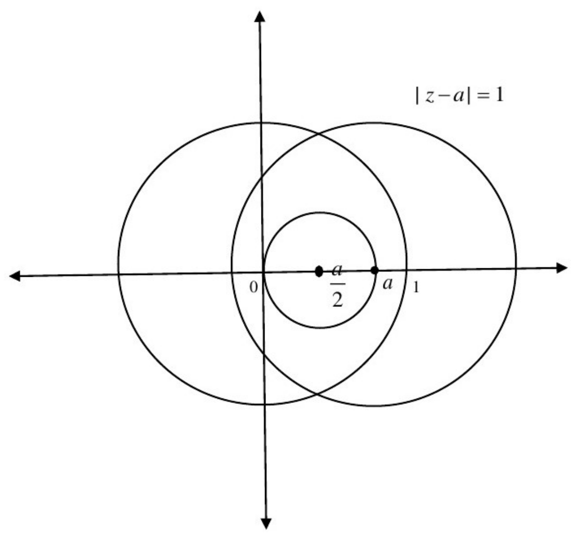

Figure 1. Geometric illustration of Theorem 2.4

Theorem 2.7. *Let* $p(z) = (z-a)^k Q(z)$, $Q(z) = \prod_{j=1}^{n-k}(z-z_j)$ *with* $0 < a \leq 1$ *and* $|z_j| \leq a$, $1 \leq j \leq n-k$. *Then at least one zero of* $p^{(m)}(z)$ $(1 \leq m \leq n-1)$ *lies in the disk*

$$\left|z - \frac{ak}{m+1}\right| \leq a\left(1 - \frac{k}{m+1}\right).$$

Proof. We have

$$p(z) = (z-a)^k Q(z),$$

where $Q(z) = \prod_{j=1}^{n-k}(z-z_j), 1 \leq j \leq n-k$. Then it is easy to see that

$$\frac{p^{(m+1)}(a)}{p^{(m)}(a)} = \frac{m+1}{m-k+1}\frac{Q^{(m-k+1)}(a)}{Q^{(m-k)}(a)} \tag{4}$$

Now $p^{(m)}(z)$ and $Q^{(m-k)}(z)$ are polynomials of degree $n-m$ and therefore, if the zeros of $p^{(m)}(z)$ are $w_1,\ w_2,\ .\ .\ .\ w_{n-m}$ and the zeros of $Q^{(m-k)}(z)$ are $\zeta_1,\ \zeta_2,\ .\ .\ .\ \zeta_{n-m}$. Then from (4) we have

$$\sum_{j=1}^{n-m}\frac{1}{a-w_j} = \frac{m+1}{m-k+1}\sum_{j=1}^{n-m}\frac{1}{a-\zeta_j} \tag{5}$$

Taking real parts on both sides of (5), we get

$$\sum_{j=1}^{n-m}\Re\frac{1}{a-w_j} = \frac{m+1}{m-k+1}\sum_{j=1}^{n-m}\Re\frac{1}{a-\zeta_j}$$

Since by Gauss-Lucas Theorem, we have $|\zeta_j| \leq a$, it follows that

$$\Re\frac{1}{a-\zeta_j} \geq \frac{1}{2a}$$

for all j. Thus,

$$\sum_{j=1}^{n-m}\Re\frac{1}{a-w_j} \geq \frac{m+1}{m-k+1}\sum_{j=1}^{n-m}\Re\frac{1}{a-\zeta_j}$$

$$\implies \qquad \frac{1}{n-m}\sum_{j=1}^{n-m}\Re\frac{1}{a-w_j} \geq \frac{1}{2a}.\frac{m+1}{m-k+1}$$

Let $\Re\frac{1}{a-w} = \max_{1\leq j\leq n-m}\Re\frac{1}{a-w_j}$, then

$$\begin{aligned}\Re\frac{1}{a-w} &\geq \frac{1}{n-m}\sum_{j=1}^{n-m}\Re\frac{1}{a-w}\\ &\geq \frac{1}{2a}.\frac{m+1}{m-k+1}\end{aligned}$$

So that,

$$\Re\frac{1}{a-w} \geq \frac{1}{2a}.\frac{m+1}{m-k+1},$$

which is equivalent to

$$\left|w - \frac{ak}{m+1}\right| \leq a\left(1 - \frac{k}{m+1}\right).$$

This completes the proof. □

Remark 2.8. The result of Goodman, Rahman and Ratti [4] for zeros on the boundary is included in Theorem 2.7 as a special case when $a = 1$, $m = 1$, $k = 1$.

Remark 2.9. For $k = 1$ and $m = 1$, we get Theorem 2.4.

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