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ROBUST SEMI-INFINITE INTERVAL-VALUED OPTIMIZATION PROBLEM WITH UNCERTAIN INEQUALITY CONSTRAINTS

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Abstract. This paper focuses on a robust semi-infinite interval-valued optimization problem with uncertain inequality constraints (RSIIVP). By employing the concept of LU-optimal solution and Extended Mangasarian-Fromovitz Constraint Qualification (EMFCQ), necessary optimality conditions are established for (RSIIVP) and then sufficient optimality conditions for (RSIIVP) are derived, by using the tools of convexity. Moreover, a Wolfe type dual problem for (RSIIVP) is formulated and usual duality results are discussed between the primal (RSIIVP) and its dual (RSIWD) problem. The presented results are demonstrated by non-trivial examples.

1. Introduction

Charnes et al. [12], developed the theory of semi-infinite programming in 1962. The class of semi-infinite programming problem is an important class of constrained optimization problem in which the number of decision variables is finite, but the number of constraints is infinite. Semi-infinite programming problem, has been the area of interest for many researchers in the recent past, as this structure spontaneously occur in many mathematical applications viz., engineering design [29], air pollution control [36], economics [37], finance [21], optimal control problems [30], robotics [17], geometry and optimization under uncertainty [3]. Some of the latest advances in semi-infinite programming can be seen in [22, 23, 35].

A mathematical programming problem under data uncertainty refers to robust optimization. In recent years, robust optimization has gained more importance due to its ability, tractability and applicability while handling the "uncertain-but bounded" non-stochastic optimization problems. The objective of the robust optimization problem is to arrive at the best solution for entities "immunized" against uncertainties. For a comprehensive study of robust optimization, the readers are advised to refer to [5–7]. Robust optimization has a wide spectrum of real-world applications, in particular, finance [15], energy [38], supply chain [28], healthcare [32], engineering [16],

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scheduling [18], machine learning [11], queueing networks [4] and revenue management [8] etc.

Interval-valued optimization problems serve as a substitute option to tackle uncertain parameters that cannot be calculated accurately. It is based on interval coefficients, which are taken as closed intervals. Many advances related to the theory of interval-valued optimization problems were analyzed in detail, readers are advised to refer [13,14]. The interval-based models have various applications in real life domains namely inventory [31], production planning [27], financial and corporate planning [25], healthcare and hospital planning [34] etc.

In the recent past, several methods have been adopted to deal with interval-valued optimization problems. Bhurjee and Panda [9] designed a framework to evaluate an efficient solution for an interval-valued optimization problem. Ahmad et al. [1] have investigated sufficient optimality conditions for LU-optimal solution with interval-valued optimization problems, based on generalized (p, r)-ρ-(η, θ)- invexity. To relate to LU-optimal solutions of primal and dual problems, they deduced Wolfe and Mond-Weir type duals, using theorems of weak, strong and strict converse duality. Later, Jayswal et al. [19] has presented a new class of interval-valued variational-like inequality problem, using the notion of LU and weakly LU-optimal solutions. In [33], Singh et al. have employed weakly continuously differentiability, to develop the necessary and sufficient optimality conditions for Karush–Kuhn–Tucker type in which, objective and constraint functions are assumed to be interval-valued. Zhang et al. [39] focused on some fundamental characterizations of an interval-valued pseudo-linear function, based on the properties of pseudo-linearity. They also obtained characterizations for the solution set of an interval-valued pseudo-linear optimization problem. Kummari and Ahmad [24] discussed sufficient optimality conditions, for non-smooth interval-valued optimization problems via L-invex-infine functions, using the concepts of LU-optimal solution. They also deduced duality results for a Wolfe type dual problem. Furthermore, sufficient optimality conditions and Mond-Weir type duality results, for an interval-valued optimization problem with vanishing constraints, related to the concepts of generalized convexity were derived by Ahmad et al. [2].

Motivated by the above developments, we dedicate this paper to analyze some new results related to robust semi-infinite interval-valued optimization problem (RSIIVP). This study is categorized in the following way: In Section 2, a few preliminary and basic notions are specified. In Section 3, we discuss robust necessary LU-optimality conditions, based on the assumption of extended Mangasarian–Fromovitz constraint qualification (EMFCQ) and derive robust sufficient LU-optimality conditions for (RSIIVP). In Section 4 we propose a duality model of Wolfe type, and duality results which hold between (RSIIVP) and its dual (RSIWD) are examined. Finally, the conclusion is presented in Section 5.

2. Preliminaries

An n-dimensional Euclidean space is R^n and its non-negative orthant is R^n_+. The set of all closed bounded intervals in R is I. Suppose $I_1 = [\eta^L, \eta^U], I_2 = [\gamma^L, \gamma^U] \in I$, then

(i) $I_1 + I_2 = \{\eta + \gamma : \eta \in I_1 \text{ and } \gamma \in I_2\} = [\eta^L + \gamma^L, \eta^U + \gamma^U]$,
(ii) $-I_1 = \{-\eta : \eta \in I_1\} = [-\eta^U, -\eta^L]$,

(iii) $I_1 - I_2 = I_1 + (-I_2) = [\eta^L - \gamma^U, \eta^U - \gamma^L]$,
(iv) $c + I_1 = \{c + \eta : \eta \in I_1\} = [c + \eta^L, c + \eta^U]$,
(v) $cI_1 = \{c\eta : \eta \in I_1\} = \begin{cases} [c\eta^L, c\eta^U], \text{ if } c \geq 0, \\ [c\eta^U, c\eta^L], \text{ if } c < 0, \end{cases}$

where c is a real number.
For $I_1 = [\eta^L, \eta^U]$ and $I_2 = [\gamma^L, \gamma^U]$, the partial ordering $\leq_{LU}$ on I is defined as $I_1 \leq_{LU} I_2$ if and only if $\eta^L \leq \gamma^L$ and $\eta^U \leq \gamma^U$. Moreover, we represent $I_1 <_{LU} I_2$ if and only if $I_1 \leq_{LU} I_2$ along with $I_1 \neq I_2$. In the other words, $I_1 <_{LU} I_2$ if and only if

$$\begin{aligned} & \eta^L < \gamma^L, \quad \eta^U < \gamma^U, \\ \text{or} \quad & \eta^L \leq \gamma^L, \quad \eta^U < \gamma^U, \\ \text{or} \quad & \eta^L < \gamma^L, \quad \eta^U \leq \gamma^U. \end{aligned}$$

Definition 2.1. A function $\zeta = [\zeta^L, \zeta^U] : R^n \to I$ is said to be convex if for all $t \in [0, 1]$,

$$\zeta[(1-t)\alpha + t\beta] \leq_{LU} (1-t)\zeta(\alpha) + t\zeta(\beta), \forall\ \alpha, \beta \in R^n. \tag{1}$$

The following example demonstrates convex functions for an interval valued functions.

Example 2.2. Let $\zeta(\alpha) = [|\alpha|, |\alpha| + 2]$, where $\alpha \in R^n$.
By (1), it follows that

$$\zeta^L[(1-t)\alpha + t\beta] \leq (1-t)\zeta^L(\alpha) + t\zeta^L(\beta), \tag{2}$$

$$\zeta^U[(1-t)\alpha + t\beta] \leq (1-t)\zeta^U(\alpha) + t\zeta^U(\beta). \tag{3}$$

Consider the following equations:

$$\zeta^L[(1-t)\alpha + t\beta] = |(1-t)\alpha + t\beta|, \tag{4}$$

$$\zeta^U[(1-t)\alpha + t\beta] = |(1-t)\alpha + t\beta| + 2. \tag{5}$$

By triangle inequality and $t, (1-t) \geq 0$, (4), (5) reduces to

$$\zeta^L[(1-t)\alpha + t\beta] \leq (1-t)|\alpha| + t|\beta|, \tag{6}$$

$$\zeta^U[(1-t)\alpha + t\beta] \leq (1-t)(|\alpha| + 2) + t(|\beta| + 2). \tag{7}$$

Therefore, from the above inequalities, ζ is a convex function on R^n.

Definition 2.3. A function $\zeta = [\zeta^L, \zeta^U] : R^n \to I$ is said to be strictly convex if for all $t \in (0, 1)$,

$$\zeta[(1-t)\alpha + t\beta] <_{LU} (1-t)\zeta(\alpha) + t\zeta(\beta), \forall\ \alpha, \beta \in R^n. \tag{8}$$

The following example demonstrates strictly convex functions for an interval valued functions.

Example 2.4. Let $\zeta(\alpha) = [\alpha^2, \alpha^2 + 2]$, where $\alpha \in R^n$.
Let α, β be such that $\alpha \neq \beta$ and $t \in (0, 1)$.
By (8), it follows that

$$\begin{aligned} &\zeta^L[(1-t)\alpha + t\beta] < (1-t)\zeta^L(\alpha) + t\zeta^L(\beta) \\ &\zeta^U[(1-t)\alpha + t\beta] < (1-t)\zeta^U(\alpha) + t\zeta^U(\beta), \end{aligned} \tag{9}$$

or

$$\begin{aligned} &\zeta^L[(1-t)\alpha + t\beta] \leq (1-t)\zeta^L(\alpha) + t\zeta^L(\beta) \\ &\zeta^U[(1-t)\alpha + t\beta] < (1-t)\zeta^U(\alpha) + t\zeta^U(\beta), \end{aligned} \tag{10}$$

or

$$\begin{aligned} &\zeta^L[(1-t)\alpha + t\beta] < (1-t)\zeta^L(\alpha) + t\zeta^L(\beta) \\ &\zeta^U[(1-t)\alpha + t\beta] \leq (1-t)\zeta^U(\alpha) + t\zeta^U(\beta). \end{aligned} \tag{11}$$

Consider the following equations:

$$\zeta^L[(1-t)\alpha + t\beta] = (1-t)^2\alpha^2 + t^2\beta^2 + 2t(1-t)\alpha\beta. \tag{12}$$

$$\zeta^U[(1-t)\alpha + t\beta] = (1-t)^2\alpha^2 + t^2\beta^2 + 2t(1-t)\alpha\beta + 2t + 2(1-t). \tag{13}$$

Since $\alpha \neq \beta$, $(\alpha - \beta)^2 > 0$. This implies,

$$\alpha^2 + \beta^2 > 2\alpha\beta. \tag{14}$$

Using (14) in (13) and (12) we obtain

$$(1-t)^2\alpha^2 + t^2\beta^2 + 2t(1-t)\alpha\beta < (1-t)^2\alpha^2 + t^2\beta^2 + t(1-t)(\alpha^2 + \beta^2),$$

$$\begin{gathered} (1-t)^2\alpha^2 + t^2\beta^2 + 2t(1-t)\alpha\beta + 2t + 2(1-t) < \\ (1-t)^2\alpha^2 + t^2\beta^2 + t(1-t)(\alpha^2 + \beta^2) + 2t + 2(1-t). \end{gathered}$$

Therefore, from the above inequalities, ζ is a strictly convex function on R^n.

Let us study the below given *semi-infinite interval-valued optimization problem* in the absences of data uncertainty:

$$\text{(SIIVP)} \qquad \begin{aligned} \min_{\alpha \in R^n} \quad & \zeta(\alpha) = [\zeta^L(\alpha), \zeta^U(\alpha)] \\ & \text{subject to} \\ & \Psi_j(\alpha) \leq 0, \ \forall\, j \in J, \end{aligned}$$

where $\zeta^L, \zeta^U : R^n \to R$ and $\Psi_j : R^n \to R$, $j \in J$ are differentiable functions with the first order partial derivatives being continuous and J is an arbitrary index set(possible infinite).
The *semi-infinite interval-valued optimization problem* with data uncertainty in the constraints as shown in the below mentioned problem:

$$\text{(USIIVP)} \quad \begin{aligned} \min_{\alpha \in R^n} \quad & \zeta(\alpha) = [\zeta^L(\alpha), \zeta^U(\alpha)] \\ & \text{subject to} \\ & \Psi_j(\alpha, \lambda_j) \leq 0, \ \forall\, j \ \in J, \end{aligned}$$

where $\zeta^L, \zeta^U : R^n \to R$ and $\Psi_j : R^n \times R^q \to R$ are differentiable functions with the first order partial derivatives being continuous and $\lambda_j \in R^q$ is an uncertain parameter which belongs to the convex compact set $\Lambda_j \subset R^q$, $j \in J$. The uncertainty set-valued function $\Lambda : J \rightrightarrows R^q$, is given $\Lambda(j) := \Lambda_j$, $\forall\ j \ \in\ J$, so,

$$\text{graph}(\Lambda) = \{(j, \lambda_j) : \lambda_j \in \Lambda_j, j \in\ J\}$$

and $\lambda \in \Lambda$ implies that λ is a choice of Λ that is, $\lambda : J \rightrightarrows R^q$ and $\lambda_j \in \Lambda_j$, $\forall\ j\ \in J$. The below mentioned problem (RSIIVP) is robust counterpart of (USIIVP):

$$(\text{RSIIVP}) \quad \min_{\alpha \in R^n} \quad \zeta(\alpha) = [\zeta^L(\alpha), \zeta^U(\alpha)]$$

$$\text{subject to}$$

$$\Psi_j(\alpha, \lambda_j) \leq 0, \forall\ \lambda_j \in \Lambda_j, \forall\ j\ \in J.$$

The robust feasible set $\mathbb{H}$ of (RSIIVP) is explained as follows:

$$\mathbb{H} = \{\alpha \in R^n : \Psi_j(\alpha, \lambda_j) \leq 0,\ \forall\ j \in J,\ \forall\ \lambda_j \in \Lambda_j\}.$$

Definition 2.5. The robust feasible point $\bar{\alpha}$ is called a robust LU-optimal solution of (RSIIVP), if there does not exists a robust feasible solution α of (RSIIVP) such that $\zeta(\alpha) <_{LU} \zeta(\bar{\alpha})$.

All through this manuscript, we assume that the following assumptions hold:

(B1) J is a compact metric space.

(B2) Λ is compact valued and upper semicontinuous on J.

(B3) $\Psi_{j_n}(\alpha_n, \lambda_{j_n}) \to \Psi_j(\alpha, \lambda_j)$, whenever $j_n \in J \to j \in J$, $\lambda_{j_n} \in \Lambda_{j_n} \to \lambda_j \in \Lambda_j$ and $\alpha_n \in R^n \to \alpha \in R^n$ as $n \to \infty$.

(B4) $\nabla\Psi_{j_n}(\alpha_n, \lambda_{j_n}) \to \nabla\Psi_j(\alpha, \lambda_j)$, whenever $j_n \in J \to j \in J$, $\lambda_{j_n} \in \Lambda_{j_n} \to \lambda_j \in \Lambda_j$ and $\alpha_n \in R^n \to \alpha \in R^n$ as $n \to \infty$.

3. Robust Necessary and Sufficient Conditions

For $\alpha \in \mathbb{H}$, we split J into two index sets.

$$J = J_1(\alpha) \cup J_2(\alpha),$$

where

$$J_1(\alpha) = \{j \in J : \exists\ \lambda_j \in \Lambda_j \text{ such that } \Psi_j(\alpha, \lambda_j) = 0\},$$
$$J_2(\alpha) = J \setminus J_1(\alpha),$$
$$\Lambda_j(\alpha) = \{\lambda_j \in \Lambda_j : \Psi_j(\alpha, \lambda_j) = 0\}.$$

Definition 3.1 (Bonnan's and Shapiro [10], Jeyakumar et al. [20]). The Extended Mangasarian-Fromovitz Constraints Qualification (EMFCQ) holds at $\alpha \in \mathbb{H}$ if and only if $d \in R^n$ such that $\nabla_\alpha \Psi_j(\alpha, \lambda_j)^T d < 0, \forall\ j \in J_1(\alpha), \forall\ \lambda_j \in \Lambda_j(\alpha)$.

Lee and Lee ([26], corollary 1), established the robust necessary optimality conditions for a weakly robust efficient solution of robust semi-infinite multi-objective optimization problem. In the view point of Lee and Lee ([26], corollary 1), if we consider $l = 2$, then we arrive at the following robust necessary LU-optimality conditions for (RSIIVP).

Theorem 3.2 (Robust Necessary LU-Optimality Conditions). *Let $\bar{\alpha}$ be a robust LU-optimal solution of (RSIIVP). Suppose that $\Psi_j(\alpha, .)$ is concave on Λ_j, for each $\alpha \in R^n$ and for each $j \in J$. Also, assume that $(B1), (B2), (B3), (B4)$ and (EMFCQ) holds at $\bar{\alpha}$. Then there exist $\bar{\rho}^L \geq 0$, $\bar{\rho}^U \geq 0$, not all zero, $(\bar{\xi}_j)_{j\in J} \in R_+^{(J)}$ and $\bar{\lambda}_j \in \Lambda_j$, $j \in J$ such that*

$$\bar{\rho}^L + \bar{\rho}^U = 1,$$

$$\bar{\rho}^L \nabla \zeta^L(\bar{\alpha}) + \bar{\rho}^U \nabla \zeta^U(\bar{\alpha}) + \sum_{j\in J} \bar{\xi}_j \nabla_\alpha \Psi_j(\bar{\alpha}, \bar{\lambda}_j) = 0,$$

and

$$\bar{\xi}_j \Psi_j(\bar{\alpha}, \bar{\lambda}_j) = 0, \; j \in J.$$

In the next theorem, we discuss robust sufficient LU-optimality conditions for (RSIIVP).

Theorem 3.3 (Robust Sufficient LU-Optimality conditions). *Let $\bar{\alpha} \in \mathbb{H}$. Suppose that $\zeta^L, \zeta^U : R^n \to R$ be convex functions. Let $\Psi_j(., \lambda_j)$ be convex on R^n, for each $\lambda_j \in \Lambda_j$ and for each $j \in J$. Suppose that there exist $\bar{\rho}^L \geq 0$, $\bar{\rho}^U \geq 0$, not all zero, $(\bar{\xi}_j)_{j\in J} \in R_+^{(J)}$ and $\bar{\lambda}_j \in \Lambda_j$, $j \in J$ such that*

$$\bar{\rho}^L \nabla \zeta^L(\bar{\alpha}) + \bar{\rho}^U \nabla \zeta^U(\bar{\alpha}) + \sum_{j\in J} \bar{\xi}_j \nabla_\alpha \Psi_j(\bar{\alpha}, \bar{\lambda}_j) = 0, \tag{15}$$

$$\bar{\xi}_j \Psi_j(\bar{\alpha}, \bar{\lambda}_j) = 0, \; j \in J. \tag{16}$$

Then $\bar{\alpha}$ is a robust LU-optimal solution of (RSIIVP).

Proof. : Suppose $\bar{\alpha}$ is not a robust LU-optimal solution of (RSIIVP), then there exists $\alpha^* \in \mathbb{H}$, such that

$$\zeta(\alpha^*) <_{LU} \zeta(\bar{\alpha}).$$

That is,

$$\zeta^L(\alpha^*) < \zeta^L(\bar{\alpha})$$
$$\zeta^U(\alpha^*) < \zeta^U(\bar{\alpha}),$$

or

$$\zeta^L(\alpha^*) \leq \zeta^L(\bar{\alpha})$$
$$\zeta^U(\alpha^*) < \zeta^U(\bar{\alpha}),$$

or

$$\zeta^L(\alpha^*) < \zeta^L(\bar{\alpha})$$
$$\zeta^U(\alpha^*) \leq \zeta^U(\bar{\alpha}).$$

Since $\bar{\rho}^L \geq 0$, $\bar{\rho}^U \geq 0$, then the above inequalities together yield

$$\bar{\rho}^L \zeta^L(\alpha^*) + \bar{\rho}^U \zeta^U(\alpha^*) < \bar{\rho}^L \zeta^L(\bar{\alpha}) + \bar{\rho}^U \zeta^U(\bar{\alpha}). \tag{17}$$

By convexity assumption of ζ^L, ζ^U and $\Psi_j(., \lambda_j)$, $j \in J$, for any $\alpha \in R^n$, we have

$$\zeta^L(\alpha) - \zeta^L(\bar{\alpha}) \geq \nabla \zeta^L(\bar{\alpha})^T(\alpha - \bar{\alpha}), \tag{18}$$

$$\zeta^U(\alpha) - \zeta^U(\bar{\alpha}) \geq \nabla \zeta^U(\bar{\alpha})^T(\alpha - \bar{\alpha}), \tag{19}$$

$$\Psi_j(\alpha, \bar{\lambda}_j) - \Psi_j(\bar{\alpha}, \bar{\lambda}_j) \geq \nabla_\alpha \Psi_j(\bar{\alpha}, \bar{\lambda}_j)^T(\alpha - \bar{\alpha}). \tag{20}$$

The inequalities (18) and (19) together with $\bar{\rho}^L \geq 0$, $\bar{\rho}^U \geq 0$ gives

(21) $\bar{\rho}^L(\zeta^L(\alpha)-\zeta^L(\bar{\alpha}))+\bar{\rho}^U(\zeta^U(\alpha)-\zeta^U(\bar{\alpha})) \geq \bar{\rho}^L\nabla\zeta^L(\bar{\alpha})^T(\alpha-\bar{\alpha})+\bar{\rho}^U\nabla\zeta^U(\bar{\alpha})^T(\alpha-\bar{\alpha}).$

Multiplying (20) by $\bar{\xi}_j \geq 0$, $j \in J$ and by feasibility of $\alpha \in \mathbb{H}$, we get

(22) $\quad 0 \geq \bar{\xi}_j\Psi_j(\alpha,\bar{\lambda}_j) = \bar{\xi}_j\Psi_j(\alpha,\bar{\lambda}_j) - \bar{\xi}_j\Psi_j(\bar{\alpha},\bar{\lambda}_j) \geq \bar{\xi}_j\nabla_\alpha\Psi_j(\bar{\alpha},\bar{\lambda}_j)^T(\alpha-\bar{\alpha}).$

By using (15), the inequality(21) implies

$$\bar{\rho}^L(\zeta^L(\alpha)-\zeta^L(\bar{\alpha}))+\bar{\rho}^U(\zeta^U(\alpha)-\zeta^U(\bar{\alpha})) \geq -\sum_{j\in J}\bar{\xi}_j\nabla_\alpha\Psi_j(\bar{\alpha},\bar{\lambda}_j)^T(\alpha-\bar{\alpha}).$$

Above inequality along with (22) gives

$$\bar{\rho}^L\zeta^L(\alpha)+\bar{\rho}^U\zeta^U(\alpha) \geq \bar{\rho}^L\zeta^L(\bar{\alpha})+\bar{\rho}^U\zeta^U(\bar{\alpha}).$$

This is contrary to (17). Thus, we can conclude the validity of the theorem. □

The following is a simple example which demonstrates Theorem 3.3.

EXAMPLE 3.4. Let us examine the uncertain semi-infinite interval-valued optimization problem.

$$\text{(USIIVP-1)} \quad \min \quad \zeta(\alpha) = [\zeta^L(\alpha), \zeta^U(\alpha)]$$

subject to

$$\Psi_j(\alpha,\bar{\lambda}_j) = j\alpha^2 - \bar{\lambda}_j\alpha \leq 0, \forall\ j \in J,$$

where $\zeta^L(\alpha) = \alpha,$ if $\alpha \geq 0$, $\zeta^U(\alpha) = \alpha^2+2,$ if $\alpha \geq 0$, and the data $\bar{\lambda}_j$ is uncertain, $\bar{\lambda}_j \in \Lambda_j = [-j+2, j+2]$ and $j \in J = [0,1]$. The robust counter part of (USIIVP-1) can be defined as follows:

$$\text{(RSIIVP-1)} \quad \min \quad \zeta(\alpha) = [\alpha, \alpha^2+2]$$

subject to

$$\Psi_j(\alpha,\bar{\lambda}_j) = j\alpha^2 - \bar{\lambda}_j\alpha \leq 0, \forall\ \bar{\lambda}_j \in \Lambda_j, \forall\ j \in J.$$

One can validate the robust feasible set is $[0,1]$. Let $\bar{\alpha} = 0$, $\bar{\rho}^L = 1/2$, $\bar{\rho}^U = 1/2$. Let $(\bar{\xi}_j)_{j\in J} \in R_+^{(J)}$ be such that $\bar{\xi}_j = 0$, $0 \leq j < 1$ and $\bar{\xi}_1 = 1/2$. Let $\bar{\lambda}_j \in [-j+2, j+2]$, $\forall\ j \in [0,1)$ and $\bar{\lambda}_1 = 1$. Clearly, $J_1(0) = [0,1]$ and $\Lambda_1(0) = \{1\}$. For $d = 1$, $\nabla_\alpha\Psi_1(0,1)^T d = -1$. So, the (EMFCQ) holds at $\bar{\alpha}$. Then, one can see that

$$\bar{\rho}^L\nabla\zeta^L(\bar{\alpha})+\bar{\rho}^U\nabla\zeta^U(\bar{\alpha})+\sum_{j\in J}\bar{\xi}_j\nabla_\alpha\Psi_j(\bar{\alpha},\bar{\lambda}_j) = 0,$$

and

$$\bar{\xi}_j\Psi_j(\bar{\alpha},\bar{\lambda}_j) = 0,\ j \in J.$$

Also, it is easy to observe that $\zeta^L(\alpha) = \alpha$, $\zeta^U(\alpha) = \alpha^2+2$ and $\Psi_j(\alpha,\bar{\lambda}_j) = j\alpha^2 - \bar{\lambda}_j\alpha \leq 0, \forall\ j \in J$, $\bar{\lambda}_j \in \Lambda_j$ are convex functions on R. Therefore, by Theorem 3.3, $\bar{\alpha} = 0$ is a robust LU-optimal solution of (RSIIVP-1).

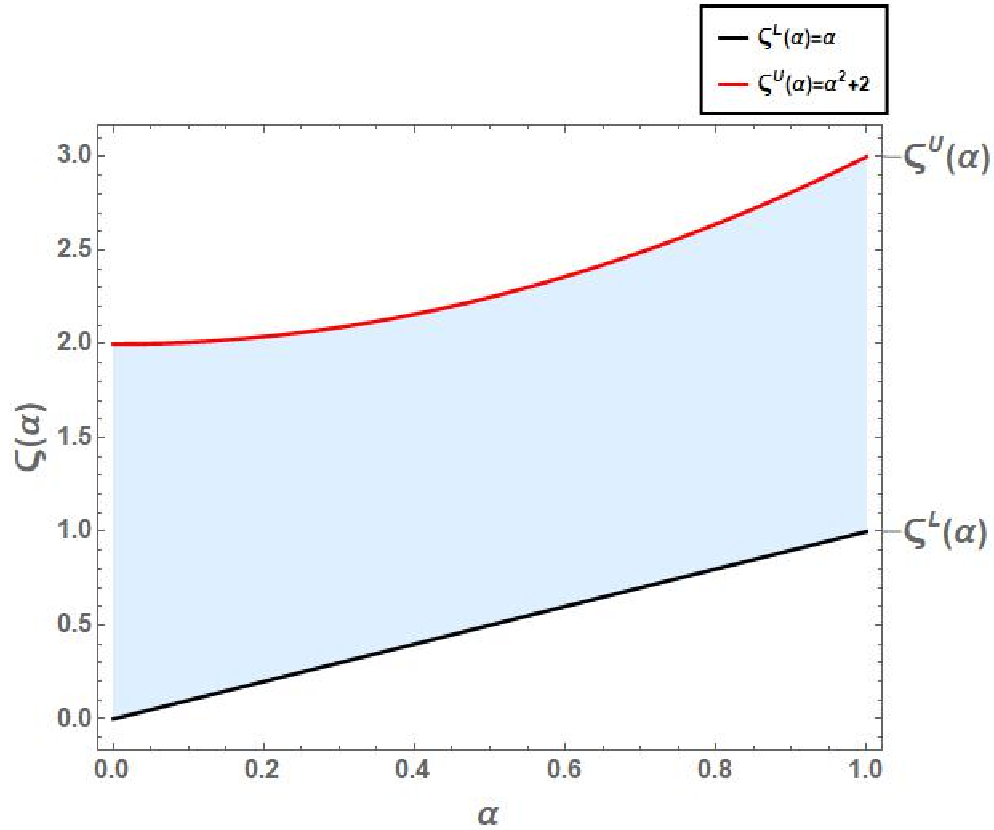

FIGURE 1. Graphical view of the objective function of the problem (USIIVP-1).

4. Wolfe Type Dual Problem

Let us examine the subsequent Wolfe type dual problem for (RSIIVP):

$$\text{(RSIWD)} \qquad \max_{(\beta,\lambda,\rho^L,\rho^U,\xi)} \zeta(\beta) = \left\{ [\zeta^L(\beta), \zeta^U(\beta)] + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j) \right\}$$

subject to

$$\rho^L \nabla \zeta^L(\beta) + \rho^U \nabla \zeta^U(\beta) + \sum_{j\in J} \xi_j \nabla_\alpha \Psi_j(\beta, \lambda_j) = 0, \tag{23}$$

$$\rho^L + \rho^U = 1, \ \rho^L, \rho^U \geq 0, \tag{24}$$

$$(\xi_j)_{j\in J} \in R_+^{(J)}, \lambda_j \in \Lambda_j, j \in J. \tag{25}$$

The robust feasible set of (RSIWD) can be represented as $\mathbb{H}_D$, which is the set of all points of the form $(\beta, \lambda, \rho^L, \rho^U, \xi) \in R^n \times \Lambda \times R_+ \times R_+ \times R_+^{(J)}$ that satisfies the constraints of dual problem.

Definition 4.1. The robust feasible point $(\bar{\beta}, \bar{\lambda}, \bar{\rho}^L, \bar{\rho}^U, \bar{\xi}) \in \mathbb{H}_D$ is called a robust LU-optimal solution of (RSIWD), if there does not exists a robust feasible solution $(\beta, \lambda, \rho^L, \rho^U, \xi)$ of (RSIWD), such that $\zeta(\bar{\beta}) + \sum_{j\in J} \bar{\xi}_j \; \Psi_j(\bar{\beta}, \bar{\lambda}_j) <_{LU} \zeta(\beta) + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j)$.

In the following part of this section, we discuss duality results between (RSIIVP) and (RSIWD).

Theorem 4.2 (Weak Duality). *Let $\zeta^L, \zeta^U : R^n \to R$ be convex functions and let $\Psi_j(., \lambda_j)$ be convex on R^n, for each $\lambda_j \in \Lambda_j$ and for each $j \in J$. Let α and $(\beta, \lambda, \rho^L, \rho^U, \xi)$ be the robust feasible solutions of (RSIIVP) and (RSIWD), respectively. Then the following inequality cannot hold:*

$$\zeta(\alpha) <_{LU} \zeta(\beta) + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j).$$

Proof. : On the contrary assume,

$$\zeta(\alpha) <_{LU} \zeta(\beta) + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j).$$

That is,

$$\zeta^L(\alpha) < \zeta^L(\beta) + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j)$$
$$\zeta^U(\alpha) < \zeta^U(\beta) + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j),$$

or

$$\zeta^L(\alpha) \leq \zeta^L(\beta) + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j)$$
$$\zeta^U(\alpha) < \zeta^U(\beta) + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j),$$

or

$$\zeta^L(\alpha) < \zeta^L(\beta) + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j)$$
$$\zeta^U(\alpha) \leq \zeta^U(\beta) + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j).$$

Since $\rho^L \geq 0$, $\rho^U \geq 0$ and $\rho^L + \rho^U = 1$, then the above inequalities together yield

$$\rho^L \zeta^L(\alpha) + \rho^U \zeta^U(\alpha) < \rho^L \zeta^L(\beta) + \rho^U \zeta^U(\beta) + \sum_{j\in J} \xi_j \Psi_j(\beta, \lambda_j). \tag{26}$$

By convexity assumption of ζ^L, ζ^U and $\Psi_j(., \lambda_j)$, $j \in J$, we have

$$\zeta^L(\alpha) - \zeta^L(\beta) \geq \nabla \zeta^L(\beta)^T(\alpha - \beta), \tag{27}$$

$$\zeta^U(\alpha) - \zeta^U(\beta) \geq \nabla \zeta^U(\beta)^T(\alpha - \beta), \tag{28}$$

$$\Psi_j(\alpha, \lambda_j) - \Psi_j(\beta, \lambda_j) \geq \nabla_\alpha \Psi_j(\beta, \lambda_j)^T(\alpha - \beta). \tag{29}$$

By inequalities (27) and (28) together with $\rho^L \geq 0$, $\rho^U \geq 0$, gives

$$\rho^L(\zeta^L(\alpha)-\zeta^L(\beta))+\rho^U(\zeta^U(\alpha)-\zeta^U(\beta)) \geq \rho^L\nabla\zeta^L(\beta)^T(\alpha-\beta)+\rho^U\nabla\zeta^U(\beta)^T(\alpha-\beta). \tag{30}$$

Multiplying (29) by $\xi_j \geq 0$, $j \in J$ and by feasibility $\alpha \in \mathbb{H}$, we get

$$-\sum_{j\in J}\xi_j\Psi_j(\beta,\lambda_j) \geq \sum_{j\in J}\xi_j\nabla_\alpha\Psi_j(\beta,\lambda_j)^T(\alpha-\beta). \tag{31}$$

On adding (30) and (31), we have

$$\rho^L(\zeta^L(\alpha)-\zeta^L(\beta))+\rho^U(\zeta^U(\alpha)-\zeta^U(\beta))-\sum_{j\in J}\xi_j\Psi_j(\beta,\lambda_j)$$
$$\geq \rho^L\nabla\zeta^L(\beta)^T(\alpha-\beta)+\rho^U\nabla\zeta^U(\beta)^T(\alpha-\beta)+\sum_{j\in J}\xi_j\nabla_\alpha\Psi_j(\beta,\lambda_j)^T(\alpha-\beta).$$

Above inequality along with (23), gives

$$\rho^L\zeta^L(\alpha)+\rho^U\zeta^U(\alpha)$$
$$\geq \rho^L\zeta^L(\beta)+\rho^U\zeta^U(\beta)+\sum_{j\in J}\xi_j\Psi_j(\beta,\lambda_j).$$

This is contrary to (26). Thus, we can conclude the validity of the theorem. □

We now re-explore Example 3.4 to demonstrate Theorem 4.2.

EXAMPLE 4.3. Consider the following uncertain semi-infinite interval-valued optimization problem:

$$\text{(USIIVP-1)} \quad \min\zeta(\alpha) = [\zeta^L(\alpha),\zeta^U(\alpha)]$$
$$\text{subject to}$$
$$\Psi_j(\alpha,\lambda_j) = j\alpha^2-\lambda_j\alpha \leq 0, \forall\ j\in J,$$

given the data uncertainty λ_j, $\lambda_j \in \Lambda_j = [-j+2, j+2]$ and $j = [0,1]$. Let ζ^L, ζ^U and Ψ_j be as in Example 3.4. The robust counter part of (USIIVP-1) is described below:

$$\text{(RSIIVP-1)} \qquad \min\zeta(\alpha) = [\alpha,\alpha^2+2]$$
$$\text{subject to}$$
$$\Psi_j(\alpha,\lambda_j) = j\alpha^2-\lambda_j\alpha \leq 0, \forall\ \lambda_j\in\Lambda_j, \forall\ j\in J.$$

The robust feasible set of (RSIIVP-1) is $\mathbb{H} = [0,1]$. Let us describe the below shown Wolfe type dual problem (RSIWD-1) for (RSIIVP-1):

$$\text{(RSIWD-1)} \qquad \max_{(\beta,\lambda,\rho^L,\rho^U,\xi)}\zeta(\beta) = \left\{[\beta,\beta^2+2]+\sum_{j\in J}\xi_j(j\beta^2-\lambda_j\beta)\right\}$$
$$\text{subject to}$$
$$\rho^L(1)+\rho^U(2\beta)+\sum_{j\in J}\xi_j(2j\beta-\lambda_j) = 0,$$
$$\rho^L+\rho^U = 1,\ \rho^L,\rho^U \geq 0,$$
$$(\xi_j)_{j\in J} \in R_+^{(J)}, \lambda_j\in\Lambda_j, j\in J.$$

Let α and $(\beta,\lambda,\rho^L,\rho^U,\xi)$ be the robust feasible solutions of (RSIIVP-1) and (RSIWD-1), respectively. Then, by the convexity of ζ^L,ζ^U and $\Psi_j(.,\lambda_j)$ on R, we see that

$$\alpha-\beta = \zeta^L(\alpha)-\zeta^L(\beta) \geq (\zeta^L)'(\beta)(\alpha-\beta) = (\alpha-\beta),$$

$$(\alpha^2+2)-(\beta^2+2)=\zeta^U(\alpha)-\zeta^U(\beta)\geq(\zeta^U)'(\beta)(\alpha-\beta)=2\beta(\alpha-\beta),$$

and for all $j\in J$,

$$\begin{aligned}(j\alpha^2-\lambda_j\alpha)-(j\beta^2-\lambda_j\beta)&=\Psi_j(\alpha,\lambda_j)-\Psi_j(\beta,\lambda_j)\\&\geq\Psi_j'(\beta,\lambda_j)(\alpha-\beta)\geq(2j\beta-\lambda_j)(\alpha-\beta).\end{aligned}$$

Let $\rho^L,\ \rho^U\geq 0$ with $\rho^L+\rho^U=1$, $(\xi_j)_{j\in J}\in R_+^{(J)}$. Then, we have

$$\rho^L(\alpha-\beta)\geq\rho^L(\alpha-\beta),$$

$$\rho^U((\alpha^2+2)-(\beta^2+2))\geq\rho^U(2\beta(\alpha-\beta)),$$

$$\sum_{j\in J}\xi_j[(j\alpha^2-\lambda_j\alpha)-(j\beta^2-\lambda_j\beta)]\geq\sum_{j\in J}\xi_j(2j\beta-\lambda_j)(\alpha-\beta).$$

Since α and $(\beta,\lambda,\rho^L,\rho^U,\xi)$ are the robust feasible solutions of (RSIIVP-1) and (RSIWD-1) respectively, $\sum_{j\in J}(j\alpha^2-\lambda_j\alpha)\leq 0$, $\rho^L(1)+\rho^U(2\beta)+\sum_{j\in J}\xi_j(2j\beta-\lambda_j)=0$. So, we obtain

$$\begin{aligned}&\rho^L\zeta^L(\alpha)+\rho^U\zeta^U(\alpha)-\left\{\rho^L\zeta^L(\beta)+\rho^U\zeta^U(\beta)+\sum_{j\in J}\xi_j\Psi_j(\beta,\lambda_j)\right\}\\&=\rho^L(\alpha-\beta)+\rho^U((\alpha^2+2)-(\beta^2+2))-\sum_{j\in J}\xi_j(j\beta^2-\lambda_j\beta)\\&\geq\rho^L(\alpha-\beta)+\rho^U(2\beta(\alpha-\beta))+\sum_{j\in J}\xi_j(j\alpha^2-\lambda_j\alpha)-\sum_{j\in J}\xi_j(j\beta^2-\lambda_j\beta)\\&\geq\rho^L(\alpha-\beta)+\rho^U(2\beta(\alpha-\beta))+\sum_{j\in J}\left\{\xi_j[(j\alpha^2-\lambda_j\alpha)-(j\beta^2-\lambda_j\beta)]\right\}\\&\geq\left\{\rho^L(1)+\rho^U(2\beta)+\sum_{j\in J}\xi_j(2j\beta-\lambda_j)\right\}(\alpha-\beta)\geq 0.\end{aligned}$$

Hence $\rho^L\zeta^L(\alpha)+\rho^U\zeta^U(\alpha)\geq\left\{\rho^L\zeta^L(\beta)+\rho^U\zeta^U(\beta)+\sum_{j\in J}\xi_j\Psi_j(\beta,\lambda_j)\right\}$. Therefore, weak duality holds.

THEOREM 4.4 (Strong Duality). *Let $\zeta^L,\zeta^U:R^n\to R$ be convex functions and let for each $\lambda_j\in\Lambda_j$ and for each $j\in J$, $\Psi_j(.,\lambda_j)$ be convex on R^n and for each $\alpha\in R^n$ and for each $j\in J$, $\Psi_j(\alpha,.)$ be concave on Λ_j. Assume that the (EMFCQ) holds at $\bar{\alpha}$. Let $\bar{\alpha}$ be a robust LU-optimal solution of (RSIIVP). Then there exists $(\bar{\rho}^L,\bar{\rho}^U,\bar{\xi},\bar{\lambda})\in R_+\times R_+\times R_+^{(J)}\times\Lambda$ such that $(\bar{\alpha},\bar{\lambda},\bar{\rho}^L,\bar{\rho}^U,\bar{\xi})$ is a robust LU-optimal solution of (RSIWD).*

Proof. : Let $\bar{\alpha}$ be a robust LU-optimal solution of (RSIIVP). Then by Theorem 3.2, there exist $\bar{\rho}^L\geq 0$, $\bar{\rho}^U\geq 0$, not all zero, $(\bar{\xi}_j)_{j\in J}\in R_+^{(J)}$ and $\bar{\lambda}_j\in\Lambda_j$, $j\in J$, such that

$$\bar{\rho}^L+\bar{\rho}^U=1,$$

$$\bar{\rho}^L\nabla\zeta^L(\bar{\alpha})+\bar{\rho}^U\nabla\zeta^U(\bar{\alpha})+\sum_{j\in J}\bar{\xi}_j\nabla_\alpha\Psi_j(\bar{\alpha},\bar{\lambda}_j)=0,$$

$$\bar{\xi}_j\Psi_j(\bar{\alpha},\bar{\lambda}_j)=0,\ j\in J. \tag{32}$$

Hence, $(\bar{\alpha}, \bar{\lambda}, \bar{\rho}^L, \bar{\rho}^U, \bar{\xi})$ is a robust feasible solution of (RSIWD). Suppose $(\bar{\alpha}, \bar{\lambda}, \bar{\rho}^L, \bar{\rho}^U, \bar{\xi})$ is not a robust LU-optimal solution of (RSIWD). Then there exists a robust feasible solution $(\bar{\beta}, \bar{\lambda}, \bar{\rho}^L, \bar{\rho}^U, \bar{\xi})$ of (RSIWD), such that

$$\zeta(\bar{\alpha}) + \sum_{j\in J} \bar{\xi}_j \Psi_j(\bar{\alpha}, \bar{\lambda}_j) <_{LU} \zeta(\bar{\beta}) + \sum_{j\in J} \bar{\xi}_j \Psi_j(\bar{\beta}, \bar{\lambda}_j). \tag{33}$$

By using (32), (33) gives

$$\zeta(\bar{\alpha}) <_{LU} \zeta(\bar{\beta}) + \sum_{j\in J} \bar{\xi}_j \Psi_j(\bar{\beta}, \bar{\lambda}_j).$$

which is contrary to the theorem of weak duality 4.2. Thus, we can conclude the validity of the theorem. □

THEOREM 4.5 (Strict Converse Duality). *Let $\bar{\alpha}$, $(\bar{\beta}, \bar{\lambda}, \bar{\rho}^L, \bar{\rho}^U, \bar{\xi})$ be the robust feasible solutions of (RSIIVP) and (RSIWD), respectively. Assume $\zeta^L, \zeta^U : R^n \to R$ are strictly convex and $\Psi_j(., \lambda_j)$ is convex on R^n, for each $\lambda_j \in \Lambda_j$, for each $j \in J$ and*

$$\bar{\rho}^L \zeta^L(\bar{\alpha}) + \bar{\rho}^U \zeta^U(\bar{\alpha}) \leq \bar{\rho}^L \zeta^L(\bar{\beta}) + \bar{\rho}^U \zeta^U(\bar{\beta}) + \sum_{j\in J} \bar{\xi}_j \Psi_j(\bar{\beta}, \bar{\lambda}_j), \tag{34}$$

then $\bar{\alpha} = \bar{\beta}$.

Proof. : On the contrary assume, $\bar{\alpha} \neq \bar{\beta}$. Since $(\bar{\beta}, \bar{\lambda}, \bar{\rho}^L, \bar{\rho}^U, \bar{\xi}) \in \mathbb{H}_D$, satisfies the relations (23) to (25). That is,

$$\bar{\rho}^L \nabla \zeta^L(\bar{\beta}) + \bar{\rho}^U \nabla \zeta^U(\bar{\beta}) + \sum_{j\in J} \bar{\xi}_j \nabla_\alpha \Psi_j(\bar{\beta}, \bar{\lambda}_j) = 0,$$

$$\bar{\rho}^L + \bar{\rho}^U = 1,\ \bar{\rho}^L, \bar{\rho}^U \geq 0,$$
$$\bar{\xi}_j \in R_+^{(J)}, j \in J, \bar{\lambda}_j \in \Lambda_j.$$

By strict convexity assumption of ζ^L, ζ^U and convexity assumption of $\Psi_j(., \bar{\lambda}_j)$, $j \in J$, we have

$$\zeta^L(\bar{\alpha}) - \zeta^L(\bar{\beta}) > \nabla \zeta^L(\bar{\beta})^T (\bar{\alpha} - \bar{\beta}), \tag{35}$$

$$\zeta^U(\bar{\alpha}) - \zeta^U(\bar{\beta}) > \nabla \zeta^U(\bar{\beta})^T (\bar{\alpha} - \bar{\beta}), \tag{36}$$

$$\Psi_j(\bar{\alpha}, \bar{\lambda}_j) - \Psi_j(\bar{\beta}, \bar{\lambda}_j) \geq \nabla_\alpha \Psi_j(\bar{\beta}, \bar{\lambda}_j)^T (\bar{\alpha} - \bar{\beta}). \tag{37}$$

By inequalities (35) and (36) together with $\bar{\rho}^L, \bar{\rho}^U \geq 0$, gives

$$\bar{\rho}^L(\zeta^L(\bar{\alpha}) - \zeta^L(\bar{\beta})) + \bar{\rho}^U(\zeta^U(\bar{\alpha}) - \zeta^U(\bar{\beta})) > \bar{\rho}^L \nabla \zeta^L(\bar{\beta})^T (\bar{\alpha} - \bar{\beta}) + \bar{\rho}^U \nabla \zeta^U(\bar{\beta})^T (\bar{\alpha} - \bar{\beta}). \tag{38}$$

Multiplying (37) by $\bar{\xi}_j \geq 0$, $j \in J$ and feasibility of $\bar{\alpha} \in \mathbb{H}$, we get

$$-\sum_{j\in J} \bar{\xi}_j \Psi_j(\bar{\beta}, \bar{\lambda}_j) \geq \sum_{j\in J} \bar{\xi}_j \nabla_\alpha \Psi_j(\bar{\beta}, \bar{\lambda}_j)^T (\bar{\alpha} - \bar{\beta}). \tag{39}$$

On adding (38) and (39), we have

$$\bar{\rho}^L(\zeta^L(\bar{\alpha}) - \zeta^L(\bar{\beta})) + \bar{\rho}^U(\zeta^U(\bar{\alpha}) - \zeta^U(\bar{\beta})) - \sum_{j\in J} \bar{\xi}_j \Psi_j(\bar{\beta}, \bar{\lambda}_j)$$

$$> \bar{\rho}^L \nabla \zeta^L(\bar{\beta})^T (\bar{\alpha} - \bar{\beta}) + \bar{\rho}^U \nabla \zeta^U(\bar{\beta})^T (\bar{\alpha} - \bar{\beta}) + \sum_{j\in J} \bar{\xi}_j \nabla_\alpha \Psi_j(\bar{\beta}, \bar{\lambda}_j)^T (\bar{\alpha} - \bar{\beta}).$$

The above inequality along with (23), implies

$$\bar{\rho}^L \zeta^L(\bar{\alpha}) + \bar{\rho}^U \zeta^U(\bar{\alpha}) > \bar{\rho}^L \zeta^L(\bar{\beta}) + \bar{\rho}^U \zeta^U(\bar{\beta}) + \sum_{j \in J} \bar{\xi}_j \Psi_j(\bar{\beta}, \bar{\lambda}_j).$$

This is contrary to (34). Therefore, the proof of the theorem stands verified. □

5. Conclusion

In the present study, sufficient optimality conditions have been generated for a semi-infinite interval-valued optimization problem with uncertain inequality constraints, by using the concept of convexity. An illustration is presented to establish the legitimacy of the sufficient optimality theorem that has been proved. In addition, the duality theorems for a Wolfe type dual problem are discussed. An example is demonstrated for the validity of weak duality theorem. It would be a good pilot study to generalize the results of this present paper to linear/non-linear semi-infinite interval-valued multi-objective optimization problem. This may be taken up as an upcoming research work for the authors.

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