

## BOUNDEDNESS OF $\mathcal{C}^{b,c}$ OPERATORS ON BLOCH SPACES

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ABSTRACT. In this article, we consider the integral operator  $\mathcal{C}^{b,c}$ , which is defined as follows:

$$\mathcal{C}^{b,c}(f)(z) = \int_0^z \frac{f(w) * F(1, 1; c; w)}{w(1-w)^{b+1-c}} dw,$$

where  $*$  denotes the Hadamard/ convolution product of power series,  $F(a, b; c; z)$  is the classical hypergeometric function with  $b, c > 0, b + 1 > c$  and  $f(0) = 0$ . We investigate the boundedness of the  $\mathcal{C}^{b,c}$  operators on Bloch spaces.

### 1. Introduction and preliminary results

Let  $\mathbb{D}$  denote the unit disc in the complex plane  $\mathbb{C}$ ,  $H(\mathbb{D})$  the set of all analytic functions on  $\mathbb{D}$  and  $\mathcal{H}_0$  be the class of all functions  $f \in H(\mathbb{D})$  with  $f(0) = 0$ .

For any complex number  $a, b, c \neq -n, n = 0, 1, 2, \dots$ , the Gaussian/classical hypergeometric function  ${}_2F_1(a, b; c; z)$  is defined by power series expansion

$${}_2F_1(a, b; c; z) = F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!} \quad (|z| < 1),$$

where  $(a, n)$  is the shifted factorial defined by Appel's symbol

$$(a, n) = a(a+1) \dots (a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, n \in \mathbb{N} = \{1, 2, \dots\}$$

and  $(a, 0) = 1$  for  $a \neq 0$ , (see [1]). Obviously,  $F(a, b; c; z)$  is an analytic function in  $\mathbb{D}$ . We refer the reader to [1] for a background on Gaussian hypergeometric functions. For the asymptotic behavior of  $F(a, b; c; z)$  for  $z$  near 1, we refer to [11] which has been used for a number of investigations.

We consider the integral operator, called  $\mathcal{C}^{b,c}$  operator for  $b, c \in \mathbb{R}, b, c > 0$  with  $b + 1 > c$ , on the space  $\mathcal{H}_0$  defined by

$$(1) \quad \mathcal{C}^{b,c}(f)(z) = \int_0^z \frac{f(w) * F(1, 1; c; w)}{w(1-w)^{b+1-c}} dw,$$

where  $*$  denotes the Hadamard/ convolution product of power series. That is, if  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $g(z) = \sum_{n=0}^{\infty} b_n z^n$  are two analytic functions in  $|z| < R$  then

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$f * g$  is defined by  $f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$  and this series converges for  $|z| < R^2$ . Moreover,

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|w|=r} f(w)g(z/w) \frac{dw}{w}, \quad |z| < rR < R^2.$$

In particular, if  $f, g$  are in  $H(\mathbb{D})$ , we have

$$(2) \quad (f * g)(rz) = \frac{1}{2\pi} \int_0^{2\pi} f(re^{it})g(ze^{-it}) dt, \quad 0 < r < 1.$$

In particular, if  $b = \beta, c = 1$  then

$$\begin{aligned} \mathcal{C}^{\beta,1}(f)(z) &= \int_0^z \frac{f(w) * F(1, 1; 1; w)}{w(1-w)^\beta} dw \\ &= \mathcal{C}_\beta(f)(z) \end{aligned}$$

which is the generalized  $\beta$ -Ces aro operator as defined in [7]. The boundedness, compactness, essential norm and spectrum of the  $\beta$ -Ces aro operators are studied by authors in [7]. Moreover, boundedness of the Ces aro and related operators in various function spaces are studied in the literature; see [5, 9, 13, 15]. In this paper, we study these operators as linear operators on  $a$ -Bloch space, denoted by  $\mathcal{B}_a$ , and is defined for each  $a > 0$  as follows:

$$\mathcal{B}_a = \{f \in H(\mathbb{D}) : \|f\|_{\mathcal{B}_a} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^a |f'(z)| < \infty\}.$$

In particular, the spaces  $\mathcal{B}_a$  becomes the classical Lipschitz and Bloch spaces whenever  $a \in (0, 1)$  and  $a = 1$  respectively.

The space  $\mathcal{B}_a$  is a complex Banach space with the norm

$$\|f\| = |f(0)| + \|f\|_{\mathcal{B}_a},$$

whereas  $\|f\|_{\mathcal{B}_a} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^a |f'(z)|$  represents a semi-norm. By restricting this space with the condition  $f(0) = 0$ , for  $f \in \mathcal{B}_a$ , we get a space, which is a subspace of  $\mathcal{B}_a$ , denoted by  $\mathcal{B}_a^0$ . The semi-norm  $\|\cdot\|_{\mathcal{B}_a}$  on  $\mathcal{B}_a$  becomes norm on  $\mathcal{B}_a^0$ . The spaces  $\mathcal{B}_a$  and  $\mathcal{B}_a^0$  together with its harmonic analog have been investigated recently by a number of authors. See for instance, see [4, 8, 10] and the references therein. Unless it is specified we consider  $a > 0$  throughout this paper. More on the Bloch space can be found in [16, 17].

Main motive of this paper is to study the boundedness properties of generalized  $\mathcal{C}^{b,c}$  operators on  $\mathcal{B}_a^0$  which include the  $\beta$ -Ces aro operators as well as the classical Ces aro operator.

## 2. Boundedness of $\mathcal{C}^{b,c}$ operators on $\mathcal{B}_a^0$

In this section, we discuss the boundedness properties of the  $\mathcal{C}^{b,c}$  operators, on  $\mathcal{B}_a^0$ . At the end of this section, we provide few examples to show that the  $\mathcal{C}^{b,c}$  operators are unbounded linear operators on  $\mathcal{B}_a^0$ , under some conditions on  $b, c$ . To obtain our desired results, we need the following lemmas.

LEMMA 1. Let  $b, c > 0$  with  $c \geq b$ . For  $f \in \mathcal{B}_a$  we have  $f * F \in \mathcal{B}_a$ , where  $F(z) = F(1, b; c; z)$ . Further

$$\|f * F\|_{\mathcal{B}_a} \leq \frac{2^a b \|f\|_{\mathcal{B}_a}}{c}.$$

*Proof.* For  $c = b$ , proof is easy as we have  $f(z) * F(1, b; b; z) = f(z)$ . Now for  $c > b$ , using the Euler integral representation with a simple calculation (see page 336 of [2]), we have

$$(3) \quad f(z) * F(1, b; c; z) = \frac{1}{B(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} f(tz) dt,$$

where  $B(b, c - b)$  is the usual beta function. Hence we get

$$\begin{aligned} (1 - |z|^2)^a |(f(z) * F(1, b; c; z))'| &\leq \frac{1}{B(b, c - b)} \int_0^1 t^b (1 - t)^{c-b-1} (1 - |z|^2)^a |f'(tz)| dt \\ &\leq \frac{1}{B(b, c - b)} \int_0^1 t^b (1 - t)^{c-b-1} (1 - t^2 |z|^2)^a |f'(tz)| dt. \end{aligned}$$

Since  $c > b$ , by taking supremum in both sides of the above inequality, we find

$$\|f * F\|_{\mathcal{B}_a} \leq \frac{2^a \|f\|_{\mathcal{B}_a}}{B(b, c - b)} \int_0^1 t^b (1 - t)^{c-b-1} dt,$$

which complete the proof. □

REMARK 2. In [2], integral representation for  $f * F(a, b; c; z)$  is given under certain conditions on the parameters  $a, b, c$  and was used to derive geometric properties of the Hadamard product. This representation may be used to generalize Lemma 1.

LEMMA 3. Let  $f \in \mathcal{B}_a$ . Suppose  $c > b > 0$  and  $F(z) = F(1, b; c; z)$ . Then we have the following properties:

(i) If  $a < 1$ , then

$$|(f * F)(z)| \leq |f(0)| + \frac{b \|f\|_{\mathcal{B}_a}}{2c(1 - a)} < \infty.$$

(ii) If  $a = 1$ , then

$$|(f * F)(z)| \leq |f(0)| + \frac{\|f\|_{\mathcal{B}_1}}{2} \log \left( \frac{1}{1 - |z|} \right).$$

(iii) If  $a > 1$ , then

$$|(f * F)(z)| \leq |f(0)| + \frac{\|f\|_{\mathcal{B}_a}}{a - 1} \left( \frac{1}{(1 - |z|)^{a-1}} - 1 \right).$$

*Proof.* Suppose  $f \in \mathcal{B}_a$  and  $z \in \mathbb{D}$ . Then

$$|(f * F)(z) - (f * F)(0)| = \left| z \int_0^1 (f * F)'(zu) du \right|.$$

Using (3) and by the definition of  $a$ -Bloch space, we have

$$\begin{aligned}
 |(f * F)(z) - (f * F)(0)| &= \frac{|z|}{B} \int_0^1 u \left( \int_0^1 t^b (1-t)^{c-b-1} |f'(tzu)| dt \right) du \\
 &\leq \frac{|z| \|f\|_{\mathcal{B}_a}}{B} \int_0^1 u \left( \int_0^1 \frac{t^b (1-t)^{c-b-1}}{(1-|z|^2 t^2 u^2)^a} dt \right) du \\
 &\leq \frac{b|z| \|f\|_{\mathcal{B}_a}}{c} \int_0^1 u \frac{1}{(1-|z|^2 u^2)^a} du \\
 (4) \qquad \qquad \qquad &= \frac{b \|f\|_{\mathcal{B}_a}}{2c} \int_{1-|z|}^1 \frac{1}{u^a} du.
 \end{aligned}$$

Since  $(f * F)(0) = f(0)$ , the above inequality gives

$$|(f * F)(z) - f(0)| \leq \frac{b \|f\|_{\mathcal{B}_a}}{2c(1-a)} \left( 1 - \frac{1}{(1-|z|)^{a-1}} \right).$$

Further, by using triangle inequality, we obtain

$$(5) \qquad |(f * F)(z)| \leq |f(0)| + \frac{b \|f\|_{\mathcal{B}_a}}{2c(1-a)} \left( 1 - \frac{1}{(1-|z|)^{a-1}} \right).$$

Let  $a < 1$ . Since  $1 - (1 - |z|)^{1-a} \leq 1$ , from (5) we obtain

$$|(f * F)(z)| \leq |f(0)| + \frac{b \|f\|_{\mathcal{B}_a}}{2c(1-a)}.$$

When  $a = 1$ , from (4), we get

$$(6) \qquad |(f * F)(z)| \leq |(f * F)(0)| + \frac{b \|f\|_{\mathcal{B}_1}}{2c} \log \left( \frac{1}{1-|z|} \right).$$

Now for  $a > 1$ , it is easily follows that

$$(7) \qquad |(f * F)(z)| \leq |f(0)| + \|f\|_{\mathcal{B}_a} \frac{1}{a-1} \left( \frac{1}{(1-|z|)^{a-1}} - 1 \right).$$

This completes the proof of this lemma. □

For  $f \in \mathcal{B}_a^0$ , Lemma 1 gives  $f * F \in \mathcal{B}_a^0$ ,  $\mathcal{C}^{b,c}(f)$  is an analytic function in  $\mathbb{D}$  and  $\mathcal{C}^{b,c}(f)(0) = 0$ . Now, we have our main result, which describes the boundedness of  $\mathcal{C}^{b,c}$  operators from  $\mathcal{B}_a^0$  to  $\mathcal{B}_a^0$  for three different restrictions on  $b$  and  $c$ . Now onward we denote  $F(z) = F(1, 1; c; z)$ .

**THEOREM 4.** *Suppose  $b, c$  are positive real numbers with  $c > 1$ . The  $\mathcal{C}^{b,c}$  operator is bounded linear operator from  $\mathcal{B}_a^0$  to  $\mathcal{B}_a^0$ , for*

- (i)  $b + 1 - c \leq a < 1$
- (ii)  $b + 1 - c \leq 1 < a$
- (iii)  $b + 1 - c < a = 1$ .

Furthermore, for  $a > 1$  or  $a < 1$ , we obtain

$$(8) \qquad \|\mathcal{C}^{b,c}(f)\|_{\mathcal{B}_a} \leq \frac{b \|f\|_{\mathcal{B}_a}}{2^{1-a} c |1-a|}.$$

*Proof.* Case(i): Suppose that  $f \in \mathcal{B}_a^0$ , for  $a < 1$ . Using (5), we have

$$\frac{|(f * F)(z)|}{|z(1-z)^{b+1-c}|} \leq \frac{\|f\|_{\mathcal{B}_a}}{|z(1-z)^{b+1-c}|} \frac{b}{2c(1-a)} \left( \frac{(1-|z|)^{a-1} - 1}{(1-|z|)^{a-1}} \right).$$

Since  $b + 1 - c \leq a < 1$  and for  $a < 1$ , we have  $1 - (1 - |z|)^{1-a} \leq |z|$ , proceeding as in the proof of Theorem 2.3 of [7], we have

$$(1 - |z|^2)^a \left| \frac{f(z) * F(z)}{z(1-z)^{b+1-c}} \right| \leq \frac{b\|f\|_{\mathcal{B}_a}}{2^{1-a}c(1-a)} (1 - |z|)^{a-b+c-1}.$$

Also  $(1 - |z|)^{a-b+c-1} < 1$  and  $z$  is arbitrary point here. Therefore (8) follows.

Case (ii): Suppose that  $f \in \mathcal{B}_a^0$ , for  $a > 1$ . From (7), we have

$$(1 - |z|^2)^a \left| \frac{f(z) * F(z)}{z(1-z)^{b+1-c}} \right| \leq \frac{(1 + |z|)^a}{|z|} \frac{b\|f\|_{\mathcal{B}_a}}{2c(a-1)} [(1 - |z|)^{c-b} - (1 - |z|)^{a-b-1+c}].$$

For  $b + 1 - c \leq 1 < a$ , this leads to

$$(1 - |z|^2)^a \left| \frac{f(z) * F(z)}{z(1-z)^{b+1-c}} \right| \leq \frac{(1 + |z|)^a}{|z|} \frac{b\|f\|_{\mathcal{B}_a}}{2c(a-1)} (1 - |z|)^{c-b} [1 - (1 - |z|)^{a-1}].$$

Since  $1 - (1 - |z|)^{a-1} \leq 1 - (1 - |z|)^{[a]}$ , where  $[\cdot]$  is a Greatest Integer Function, we have

$$(1 - |z|^2)^a \left| \frac{f(z) * F(z)}{z(1-z)^{b+1-c}} \right| \leq \frac{2^{a-1}}{|z|} \frac{b\|f\|_{\mathcal{B}_a}}{c(a-1)} (1 - |z|)^{c-b} [1 - (1 - |z|)^{[a]}].$$

Since  $(1 - |z|)^{-[a]} = F([a]; 1; 1; |z|)$ , a simple calculation gives

$$1 - (1 - |z|)^{[a]} = [a] F(-[a] + 1; 1; 2; |z|).$$

Thus, as  $b < c$  we obtain

$$(1 - |z|^2)^a \left| \frac{f(z) * F(z)}{z(1-z)^{b+1-c}} \right| \leq \frac{2^{a-1}[a]b\|f\|_{\mathcal{B}_a}}{c(a-1)} F(-[a] + 1; 1; 2; |z|).$$

We use the formula for  $c > b - m$ ,  $F(-m; b; c; 1) = \frac{(c-b)_m}{(c)_m}$ . Since  $z$  is arbitrary point in  $\mathbb{D}$ , therefore we have

$$\|\mathcal{C}^{b,c}(f)\|_{\mathcal{B}_a} \leq \frac{2^{a-1}[a]\|f\|_{\mathcal{B}_a}}{c(a-1)} \sup F(-[a] + 1; 1; 2; |z|) = \frac{b2^{a-1}\|f\|_{\mathcal{B}_a}}{c(a-1)}$$

which completes the proof.

Case (iii) Suppose that  $f * F \in \mathcal{B}_a^0$ , for  $a = 1$ . From (6), we have

$$\frac{|(f * F)(z)|}{|z(1-z)^{b+1-c}|} \leq \frac{\|f\|_{\mathcal{B}_1}}{2|z(1-z)^{b+1-c}|} \log \left( \frac{1}{1 - |z|} \right).$$

From this we get

$$(1 - |z|^2)^a \left| \frac{f(z) * F(z)}{z(1-z)^{b+1-c}} \right| \leq \frac{(1 - |z|^2)^{c-b}}{|z|} \frac{\|f\|_{\mathcal{B}_1}}{2} \log \left( \frac{1}{1 - |z|} \right).$$

For  $b + 1 - c < a = 1$ , Since  $z$  is arbitrary point, we have

$$\|C^{b,c}(f)\|_{\mathcal{B}_1} \leq \sup \left\{ \frac{(1 - |z|)^{c-b}}{|z|} \log \left( \frac{1}{1 - |z|} \right) : z \in \mathbb{D} \right\} \|f\|_{\mathcal{B}_1}.$$

This completes the proof. □

**Counterexamples.** We just proved that either of the cases  $b + 1 - c \leq a < 1$  or  $b + 1 - c \leq 1 < a$  or  $b + 1 - c < a = 1$ , the  $C^{b,c}$  operators are bounded from  $\mathcal{B}_a^0$  to  $\mathcal{B}_a^0$ . We now show that for the remaining cases:  $b + 1 - c > a$  or  $b + 1 - c = a \geq 1$  and  $1 < b + 1 - c < a$ , the  $C^{b,c}$  operators need not be bounded as the following counterexamples show.

EXAMPLE 5. Let  $f(z) = z^k, k \geq 1$ . Then  $f * F \in \mathcal{B}_a^0$ , for  $b + 1 - c > a$  since

$$(1 - |z|^2)^a \left| \frac{Kz^k}{z(1 - z)^{b+1-c}} \right| = K(1 + |z|^2)^a \frac{|z|^{k-1}(1 - |z|)^a}{|(1 - z)^{b+1-c}|},$$

where  $K = \frac{(b, k)}{(1, k)} > 0$ . For  $z = t \in (0, 1)$ , we obtain

$$(1 - |z|^2)^a \left| \frac{Kz^k}{z(1 - z)^{b+1-c}} \right| = K(1 + t)^a \frac{t^{k-1}(1 - t)^a}{(1 - t)^{b+1-c}} = K \frac{t^{k-1}(1 + t)^a}{(1 - t)^{b+1-c-a}}.$$

As  $t$  tends to 1, the right hand side term tends to  $\infty$ . Therefore, the  $C^{b,c}$  operator is an unbounded linear operator from  $\mathcal{B}_a^0$  to  $\mathcal{B}_a^0$ , for  $b + 1 - c > a$ .

EXAMPLE 6. Let  $f(z) = \log(1 - z)$ , where the principal value of the branch of logarithm is chosen. Using (3), when  $c > 1, f * F \in \mathcal{B}_a^0$ , for  $a \geq 1$  and for  $z = u \in (0, 1)$ , we have

$$\begin{aligned} (1 - |z|^2)^a \left| \frac{\log(1 - z) * F(1, 1; c; z)}{z(1 - z)^{b+1-c}} \right| &= (1 - u^2)^a \left| \frac{\int_0^1 t^{b-1}(1 - t)^{c-1-1} \log(1 - tu) dt}{B(b, c - b)u(1 - u)^{b+1-c}} \right| \\ &= \frac{(1 + u)^a}{(1 - u)^{b+1-c-a}} \left| \frac{\int_0^1 t^{b-1}(1 - t)^{c-1-1} \log(1 - tu) dt}{B(b, c - b)u} \right| \end{aligned}$$

Since the integral on the right hand side is finite so, for  $b + 1 - c \geq a$ , as  $u$  tends to 1, the right hand side term diverges to  $\infty$ . Therefore, the  $C^{b,c}$  operator is an unbounded linear operator from  $\mathcal{B}_a^0$  to  $\mathcal{B}_a^0$ , for  $b + 1 - c \geq a \geq 1$ .

EXAMPLE 7. Let  $f(z) = \frac{z}{(1 - z)^a}$ , for  $a > 0$ . Then  $f * F \in \mathcal{B}_{a+1}^0$ . Using (3), when  $c > 1$ , and for  $z = u \in (0, 1)$ , we have

$$\begin{aligned} (1 - |z|^2)^{a+1} \left| \frac{f(z) * F(1, 1; c; z)}{z(1 - z)^{b+1-c+a}} \right| &= (1 - u^2)^a \left| \frac{\int_0^1 t^{b-1}(1 - t)^{c-1-1} \frac{tu}{(1 - tu)^a} dt}{B(b, c - b)u(1 - u)^{b+1-c}} \right| \\ &= \frac{(1 - u^2)^{a+1}}{(1 - u)^{b+1-c}} \int_0^1 t^b(1 - t)^{c-1-1}(1 - tu)^{-a} dt \\ &> \frac{(1 - u^2)^{a+1}}{(1 - u)^{b+1-c}} \int_0^1 t^b(1 - t)^{c-1-1} dt. \end{aligned}$$

For  $z = t \in (0, 1)$ , then it yields

$$(1 - |z|^2)^{a+1} \left| \frac{z}{z(1-z)^{b+1-c+a}} \right| = (1+t)^{a+1} \frac{(1-t)^{a+1}}{(1-t)^{b+1-c+a}} = \frac{(1+t)^{a+1}}{(1-t)^{b-c}}.$$

Then for  $b+1-c > 1$ , and  $c > a+1$  as  $t$  tends to 1, the right hand side term diverges to  $\infty$ . Therefore, the  $\mathcal{C}^{b,c}$  operator is an unbounded linear operator from  $\mathcal{B}_a^0$  to  $\mathcal{B}_a^0$ , for  $b+1-c > 1$ .

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