

SOME PROPERTIES OF SCHENSTED ALGORITHM USING VIENNOT'S GEOMETRIC INTERPRETATION

JAEJIN LEE

ABSTRACT. Schensted algorithm was first described in 1938 by Robinson [5], in a paper dealing with an attempt to prove the correctness of the Littlewood-Richardson rule. Schensted [9] rediscovered Schensted algorithm independently in 1961 and Viennot [12] gave a geometric interpretation for Schensted algorithm in 1977. In this paper we describe some properties of Schensted algorithm using Viennot's geometric interpretation.

1. Introduction

Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ be a partition of the nonnegative integer n , denoted $\lambda \vdash n$, so λ is a weakly decreasing sequence of positive integers summing to n . We will also let λ stand for the Ferrers diagram D_λ of λ written in English notation with λ_i nodes or cells in the i th row from the top.

Given $\lambda \vdash n$, a *standard Young tableau* T of shape λ is a filling of the diagram D_λ with positive integers $1, 2, \dots, n$ such that rows and columns strictly increase. For example,

$$\begin{array}{ccc} 1 & 3 & 5 \\ 2 & 6 & \\ 4 & 7 & \end{array}$$

Received July 4, 2013. Revised August 1, 2013. Accepted August 2, 2013.
2010 Mathematics Subject Classification: 05E10.

Key words and phrases: partition, standard Young tableau, P -tableau, Q -tableau, Schensted algorithm.

© The Kangwon-Kyungki Mathematical Society, 2013.

This is an Open Access article distributed under the terms of the Creative Commons Attribution Non-Commercial License (<http://creativecommons.org/licenses/by-nc/3.0/>) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

where $a = 10$, $b = 11$, $c = 12$, $d = 13$ and $e = 14$.

In this paper we describe the ways to find P -tableaux and Q -tableaux of permutations π^r , π^* and $\pi^\#$ without using Schützenberger’s evacuation algorithm.

Section 2 gives Viennot’s geometric interpretation for Schensted algorithm. In Section 3 we describe the ways to find P -tableaux and Q -tableaux of permutations π^r , π^* and $\pi^\#$ using Viennot’s geometric interpretation for Schensted algorithm.

2. Geometric interpretation for Schensted algorithm

In this section we describe Viennot’s geometric interpretation for Schensted algorithm. See [12] or [7] for further exposition.

EXAMPLE 2.1. Let

$$\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7 \in S_7$$

- Consider the first quadrant of the Cartesian plane. Given a permutation $\pi = x_1x_2 \cdots x_n$, represent x_i by a box with coordinates (i, x_i) . See the figure 1.
- Imagine a light shining from the origin so that each box casts a shadow with boundaries parallel to the coordinate axes. The shadow cast by the box at $(4, 6)$ looks like the figure 2.
- Consider those points of the permutation that are in the shadow of no other point. In this case $(1, 4)$, $(2, 2)$, and $(6, 1)$. The first shadow line, L_1 , is the boundary of the combined shadows of these boxes. In the figure 3, the appropriate line has been thickened. Note that this is a broken line consisting of line segments and exactly one horizontal and one vertical ray. To form the second shadow line, L_2 , one removes the boxes on the first shadow line and repeats this procedure.

Given a permutation displayed in the plane, we form its *shadow lines* L_1, L_2, \dots as follows. Assuming that L_1, \dots, L_{i-1} have been constructed, remove all boxes on these lines. Then L_i is the boundary of the shadow of the remaining boxes. The x -coordinate of L_i is

$$x_{L_i} = \text{the } x\text{-coordinate of } L_i\text{'s vertical ray}$$

and the y -coordinate is

$$y_{L_i} = \text{the } y\text{-coordinate of } L_i\text{'s horizontal ray}$$

The shadow lines make up the *shadow diagram* of π .

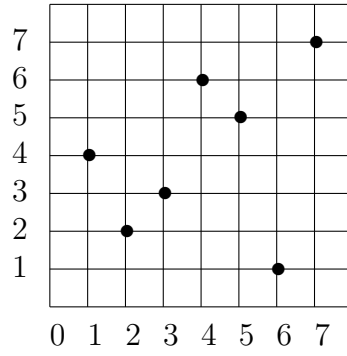


FIGURE 1. The coordinates (i, x_i) of π .

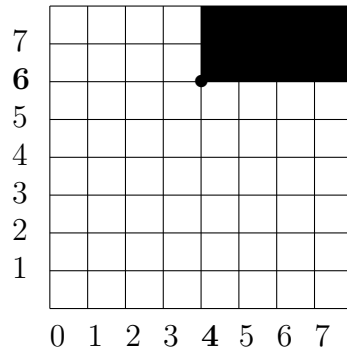


FIGURE 2. The shadow at $(4,6)$.

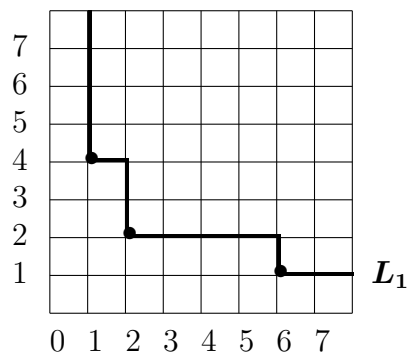


FIGURE 3. The first shadow line, L_1 .

EXAMPLE 2.2. In the previous example, there are four shadow lines, and their x - and y -coordinates are shown above and to the left of the figure 4, respectively.

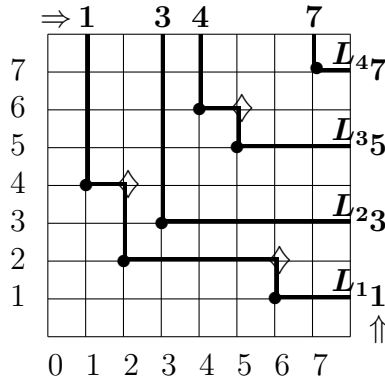


FIGURE 4. Four shadow lines for $\pi = 4236517$

Compare the coordinates of our shadow lines with the first rows of the tableaux

$$P(\pi) = \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 6 & & \\ 4 & & & \end{matrix} \quad \text{and} \quad Q(\pi) = \begin{matrix} 1 & 3 & 4 & 7 \\ 2 & 5 & & \\ 6 & & & \end{matrix}$$

computed by Schensted Algorithm. It seems as if

$$P_{1,j} = y_{L_j} \quad \text{and} \quad Q_{1,j} = x_{L_j}$$

for all j .

In fact, even more is true. The boxes on line L_j are precisely those elements passing through the $(1, j)$ cell during the construction of P , as the next result shows.

LEMMA 2.3. *Let the shadow diagram of $\pi = x_1x_2 \cdots x_n$ be constructed as before. Suppose the vertical line $x = k$ intersects i of the shadow lines. Let y_j be the y -coordinate of the lowest point of the intersection with L_j . Then the first row of the $P_k = P(x_1 \dots x_k)$ is*

$$(1) \quad R_1 = y_1y_2 \cdots y_i$$

Proof. : See [7]. □

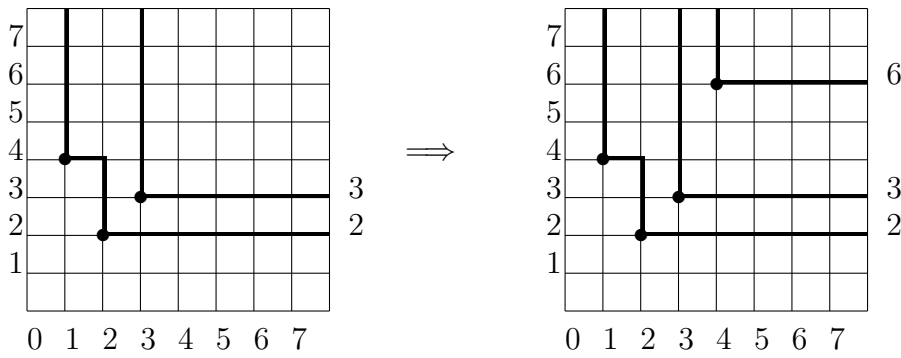
EXAMPLE 2.4. Let $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$, then

$$P_0 = \emptyset \quad P_1 = 4 \quad P_2 = \begin{matrix} 2 \\ 4 \end{matrix} \quad P_3 = \begin{matrix} 2 & 3 \\ 4 \end{matrix}$$

$$P_4 = \begin{matrix} 2 & 3 & 6 \\ 4 \end{matrix} \quad P_5 = \begin{matrix} 2 & 3 & 5 \\ 4 & 6 \end{matrix} \quad P_6 = \begin{matrix} 1 & 3 & 5 \\ 2 & 6 \\ 4 \end{matrix} \quad P_7 = \begin{matrix} 1 & 3 & 5 & 7 \\ 2 & 6 \\ 4 \end{matrix}$$

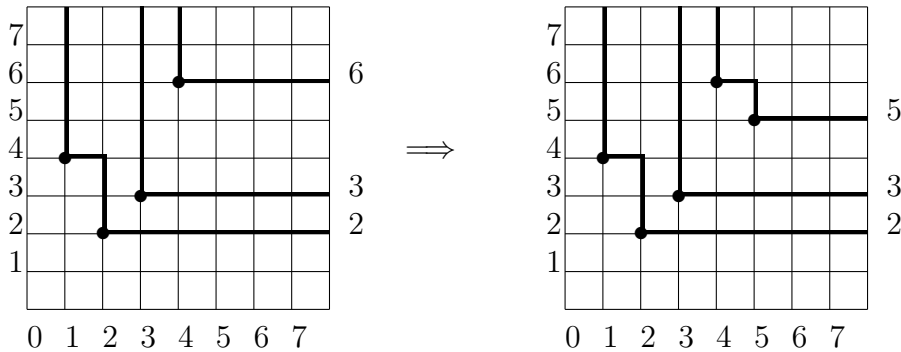
(Case 1) $k = 3$, $P_3 = P(x_1x_2x_3) = P(423) = \begin{matrix} 2 & 3 \\ 4 \end{matrix}$, $R_1 = y_1y_2$

Then $x_{k+1} = x_4 = 6 > 3 = y_2$ and so $(k + 1, x_{k+1}) = (4, 6)$ starts a new shadow line. Hence, $y_3 = x_4 = 6$.



(Case 2) $k = 4$, $P_4 = P(4236) = \begin{matrix} 2 & 3 & 6 \\ 4 \end{matrix}$, $R_1 = 236 = y_1y_2y_3$

Then $y_1 < \dots < y_{j-1} < x_5 = 5 < y_j < \dots < y_3 = 6$ and so $(k + 1, x_{k+1}) = (5, 5)$ is added to line L_j . Hence, $y'_j = x_{k+1} = x_5 = 5$.



It follows from the previous lemma that the shadow diagram of π can be read left to right like a time-line recording the construction of $P(\pi)$. At the k -th stage, the line $x = k$ intersects one shadow line in a ray or line segment and all the rest in single points. In terms of the first row of P_k : a ray corresponds to placing an element at the end, a line segment corresponds to displacing an element, and the points correspond to elements that are unchanged.

COROLLARY 2.5. *If the permutation π has Schensted tableaux (P, Q) and shadow lines L_j , then, for all j ,*

$$P_{1,j} = y_{L_j} \quad \text{and} \quad Q_{1,j} = x_{L_j}.$$

Proof. See [7]. □

EXAMPLE 2.6. Let

$$\pi = 4 \ 2 \ 3 \ 6 \ 5 \ 1 \ 7 \in S_7$$

Then the first, second and third rows come from the thickened and dashed lines, respectively, of the figure 5, 6 and 7.

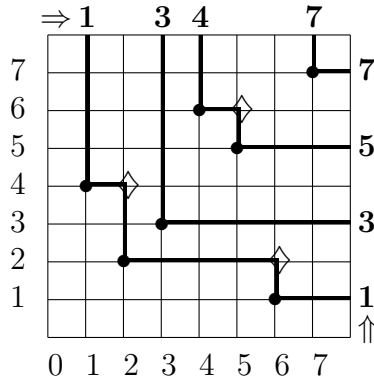


FIGURE 5. The first row of P and Q .

The i -th skeleton of $\pi \in S_n$, $\pi^{(i)}$, is defined inductively by $\pi^{(1)} = \pi$ and

$$\pi^{(i)} = \begin{matrix} k_1 & k_2 & \cdots & k_m \\ l_1 & l_2 & \cdots & l_m \end{matrix}$$

where $(k_1, l_1), \dots, (k_m, l_m)$ are the NorthEast corners of the shadow diagram of $\pi^{(i-1)}$ listed in lexicographic order. The shadow lines for $\pi^{(i)}$ are denoted $L_j^{(i)}$.

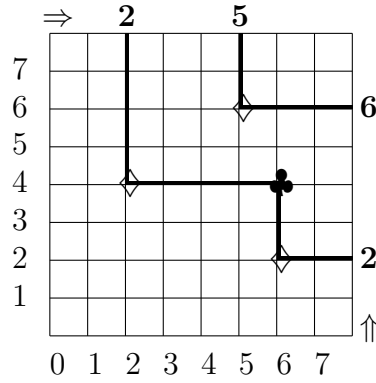


FIGURE 6. The second row of P and Q .

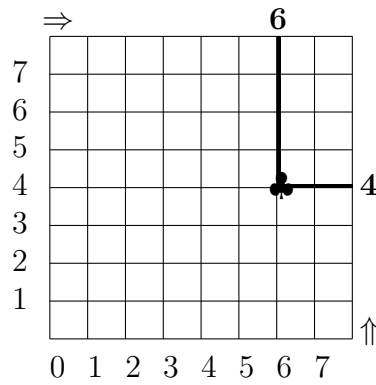


FIGURE 7. The third row of P and Q .

EXAMPLE 2.7. Let $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$ and

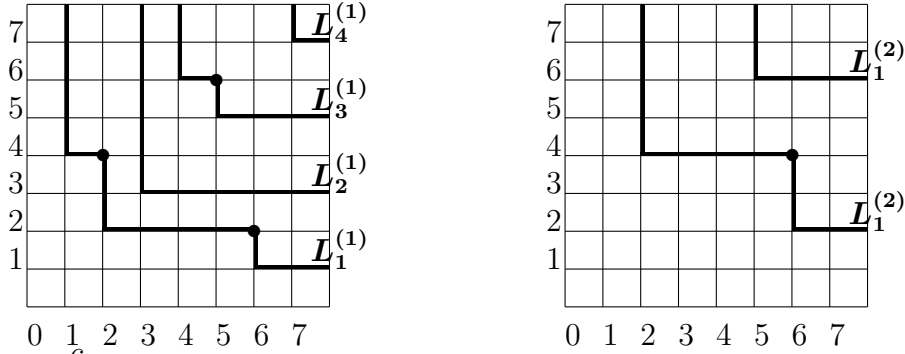
$$(P, Q) = \begin{pmatrix} 1 & 3 & 5 & 7 & 1 & 3 & 4 & 7 \\ 2 & 6 & & & 2 & 5 & & \\ 4 & & & & 6 & & & \end{pmatrix}.$$

$\pi^{(2)} = \begin{matrix} 2 & 5 & 6 \\ 4 & 6 & 2 \end{matrix}$, where $\{4, 6, 2\}$ and $\{2, 5, 6\}$ are the remainder except

for the first row of P and Q , respectively.

$$P_{1,j} = y_{L_j^{(1)}} = y_{L_1^{(1)}}y_{L_2^{(1)}}y_{L_3^{(1)}}y_{L_4^{(1)}} = 1\ 3\ 5\ 7$$

$$Q_{1,j} = x_{L_j^{(1)}} = x_{L_1^{(1)}}x_{L_2^{(1)}}x_{L_3^{(1)}}x_{L_4^{(1)}} = 1\ 3\ 4\ 7$$



$\pi^{(3)} = \begin{smallmatrix} 6 \\ 4 \end{smallmatrix}$, where 4 and 6 are the remainder except for the first, second rows of P and Q , respectively.

$P_{2,j} = y_{L_j^{(2)}} = y_{L_1^{(2)}}y_{L_2^{(2)}} = 2\ 6$, $Q_{2,j} = x_{L_j^{(2)}} = x_{L_1^{(2)}}x_{L_2^{(2)}} = 2\ 5$
 Continuing this processing, $P_{3,j} = y_{L_1^{(3)}} = 4$, $Q_{3,j} = x_{L_1^{(3)}} = 6$.

THEOREM 2.8. Suppose $\pi \rightarrow (P, Q)$. Then $\pi^{(i)}$ is a partial permutation such that

$$\pi^{(i)} \rightarrow (P^{(i)}, Q^{(i)})$$

where $P^{(i)}$ (respectively, $Q^{(i)}$) consists of the rows i and below of P (respectively, Q). Furthermore,

$$P_{i,j} = y_{L_j^{(i)}} \quad \text{and} \quad Q_{i,j} = x_{L_j^{(i)}}$$

for all i, j .

Proof. See [7]. □

3. Main results

Given a permutation $\pi = x_1x_2 \cdots x_{n-1}x_n \in S_n$, we define new permutations π^r , π^* and $\pi^\#$ as follows.

$$\begin{aligned} \pi^r &= x_nx_{n-1} \cdots x_2x_1 \\ \pi^* &= (n+1-x_1)(n+1-x_2) \cdots (n+1-x_{n-1})(n+1-x_n) \\ \pi^\# &= (n+1-x_n)(n+1-x_{n-1}) \cdots (n+1-x_2)(n+1-x_1). \end{aligned}$$

Note that $(\pi^*)^r = \pi^\#$. Until now we used Schützenberger’s evacuation algorithm to compute P -tableaux and Q -tableaux of permutations

π^r , π^* and $\pi^\#$. See [7] for detail. In this section we describe the ways to find P -tableaux and Q -tableaux of permutations π^r , π^* and $\pi^\#$ directly from Viennot’s geometric interpretation for Schensted algorithm.

Given a permutation $\pi \in S_n$, let $\pi \xleftrightarrow{[S]} (P, Q)$. Then we can find P -tableaux and Q -tableaux of permutations π^r , π^* and $\pi^\#$ as the following propositions.

PROPOSITION 3.1. *Let $\pi^* \xleftrightarrow{[S]} (P^*, Q^*)$. Then (P^*, Q^*) can be obtained as follows:*

1. *Imagine a light shining from $(0, n + 1)$ and get a new shadow diagram of π .*
2. *Change any coordinate y_{L_i} to $(n + 1) - y_{L_i}$.*
3. *P -tableaux and Q -tableaux are obtained similarly as if we read the original shadow diagram of π .*

EXAMPLE 3.2. Let $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$. Then,

$$\pi \xleftrightarrow{[S]} (P, Q) = \left(\begin{array}{cccc|cccc} 1 & 3 & 5 & 7 & 1 & 3 & 4 & 7 \\ 2 & 6 & & & 2 & 5 & & \\ 4 & & & & & 6 & & \end{array} \right).$$

By the reading coordinate steps of Figure 8 and 9, we obtain that

$$(P^*, Q^*) = \left(\begin{array}{cccc|cccc} 1 & 3 & 7 & 1 & 2 & 6 & & \\ 2 & 5 & & 3 & 5 & & & \\ 4 & & & 4 & & & & \\ 6 & & & 7 & & & & \end{array} \right)$$

Note that $P^* = (\text{ev}(P))^t$ and $Q^* = Q^t$.

PROPOSITION 3.3. *Let $\pi^r \xleftrightarrow{[S]} (P^r, Q^r)$. Then (P^r, Q^r) can be obtained as follows:*

1. *Imagine a light shining from $(n + 1, 0)$ and get a new shadow diagram of π .*
2. *Change any coordinate x_{L_i} to $(n + 1) - x_{L_i}$.*
3. *P -tableaux and Q -tableaux are obtained similarly as if we read the original shadow diagram of π .*

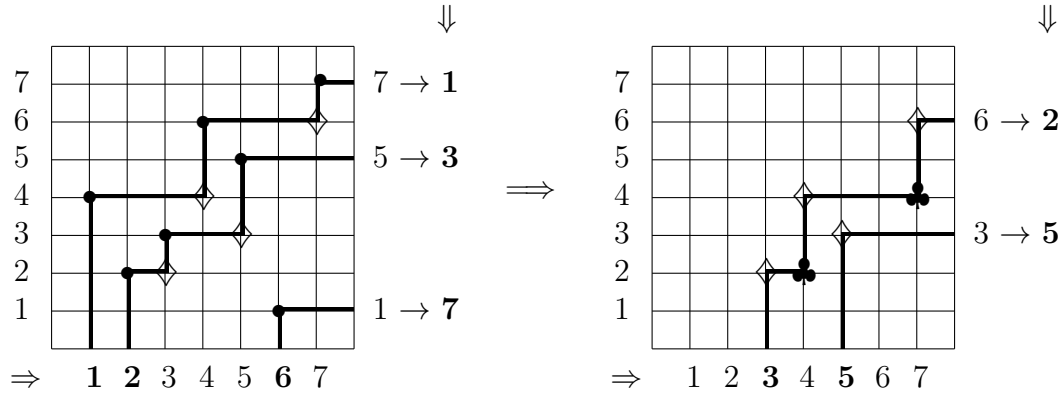


FIGURE 8. First and Second rows for π^*

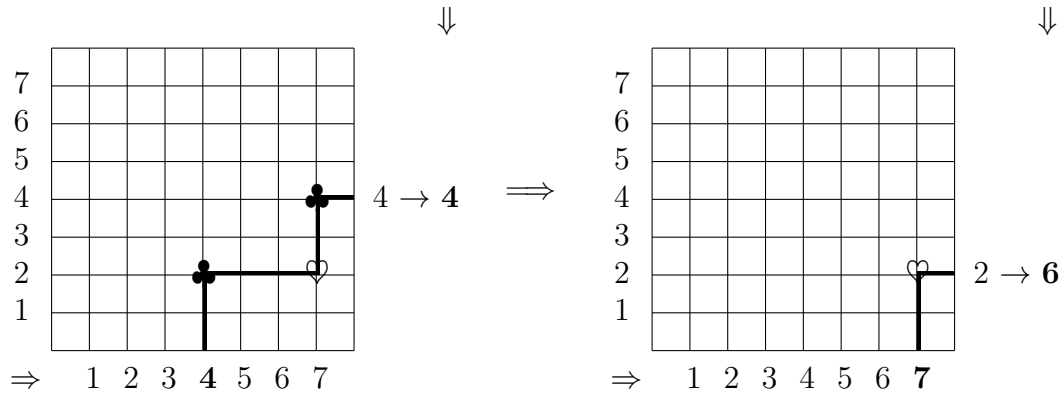


FIGURE 9. Third and Fourth rows for π^*

EXAMPLE 3.4. Let $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$. By the reading coordinate steps of Figure 10 and 11, we obtain that

$$(P^r, Q^r) = \begin{pmatrix} 1 & 2 & 4 & 1 & 3 & 4 \\ 3 & 6 & & 2 & 7 \\ 5 & & & 5 \\ 7 & & & 6 \end{pmatrix}$$

Note that $P^r = P^t$ and $Q^r = (\text{ev}(Q))^t$.

PROPOSITION 3.5. Let $\pi^\# \xleftrightarrow{[S]} (P^\#, Q^\#)$. Then $(P^\#, Q^\#)$ can be obtained as follows:

1. Imagine a light shining from $(n + 1, n + 1)$ and get a new shadow diagram of π .

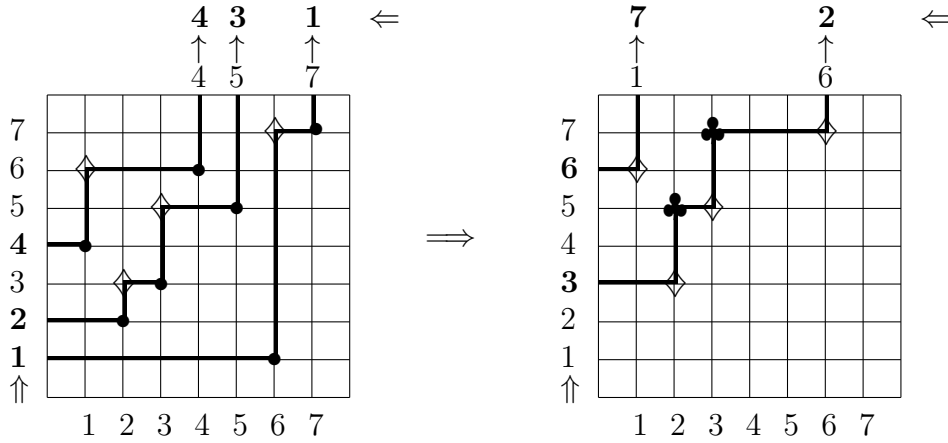


FIGURE 10. First and Second rows for π^r

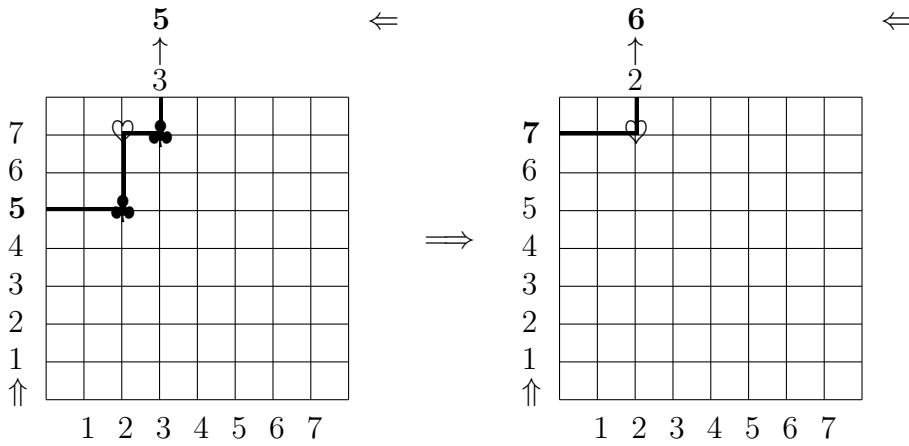


FIGURE 11. Third and Fourth rows for π^r

2. Change any coordinate x_{L_j} and y_{L_i} to $(n + 1) - x_{L_j}$ and $(n + 1) - y_{L_i}$, respectively.
3. P -tableaux and Q -tableaux are obtained similarly as if we read the original shadow diagram of π .

EXAMPLE 3.6. Let $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$. By the reading coordinate steps of Figure 12 and 13, we obtain that

$$(P^\sharp, Q^\sharp) = \left(\begin{array}{cccccc} 1 & 2 & 4 & 6 & 1 & 2 & 5 & 6 \\ 3 & 5 & & & 3 & 7 & & \\ 7 & & & & & & 4 & \end{array} \right)$$

Note that $P^\# = \text{ev}(P)$ and $Q^\# = \text{ev}(Q)$.

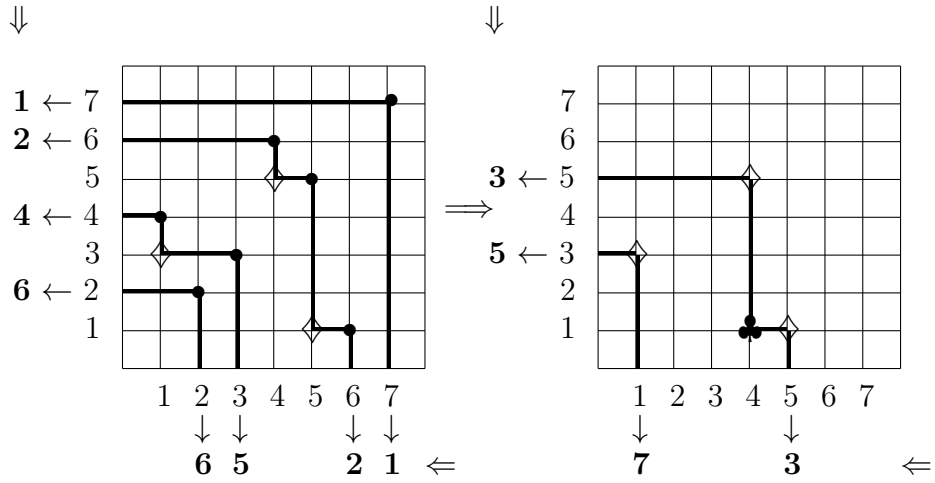


FIGURE 12. First and Second rows for $\pi^\#$

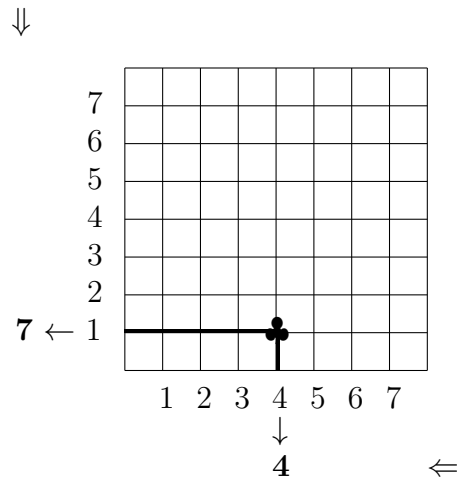


FIGURE 13. Third row for $\pi^\#$

References

- [1] A. Berele, *A Schensted-type correspondence for the symplectic group*, J. Combin. Theory Ser. A **43** (1986), 320–328.

- [2] C. Greene, *An extension of Schensted's theorem*, Adv. in Math., **14** (1974), 254–265.
- [3] J. J. Lee, *A Schensted algorithm for shifted rim hook tableaux*, Korean J. Math., **31** (1994), 179–203.
- [4] D. E. Knuth, *Permutations, matrices and generalized Young tableaux*, Pacific J. Math., **34** (1970), 709–727.
- [5] G. de B. Robinson, *On the representations of the symmetric group*, Amer. J. Math., **60**, (1938), 745–760.
- [6] B. E. Sagan, *Shifted tableaux, Schur Q -functions and a conjecture of R. Stanley*, J. Combin. Theory Ser. A **45** (1987), 62–103.
- [7] B. E. Sagan, *The symmetric group*, Springer-Verlag, New York, 2000.
- [8] B. E. Sagan and R. P. Stanley, *Robinson-Schensted algorithms for skew tableaux*, J. Combin. Theory Ser. A, **55** (1990), 161–193.
- [9] C. Schensted, *Longest increasing and decreasing subsequences*, Canad. J. Math., **13** (1961), 179–191.
- [10] D. W. Stanton and D. E. White, *A Schensted algorithm for rim hook tableaux*, J. Combin. Theory Ser. A **40** (1985), 211–247.
- [11] M. P. Schützenberger, *Quelques remarques sur une construction de Schensted*, Math., Scand. **12** (1963), 117–128.
- [12] G. Viennot, *Une forme géométrique de la correspondance de Robinson-Schensted*, in *Combiatoire et Représentation du Groupe Symétrique*, D. Foata ed., Lecture Notes in Math., Vol. 579, Springer-Verlag, New York, NY, 1977, 29–58.

Department of Mathematics
Hallym University
Chunchon, Korea 200–702
E-mail: jjlee@hallym.ac.kr