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A FINITE ADDITIVE SET OF IDEMPOTENTS IN RINGS

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Abstract. Abstract. Let R be a ring with identity 1, $I(R) \neq \{0\}$ be the set of all nonunit idempotents in R, and $M(R)$ be the set of all primitive idempotents and 0 of R. We say that $I(R)$ is *additive* if for all $e, f \in I(R)$ $(e \neq f)$, $e+f \in I(R)$. In this paper, the following are shown: (1) $I(R)$ is a finite additive set if and only if $M(R) \setminus \{0\}$ is a complete set of primitive central idempotents, $\mathrm{char}(R) = 2$ and every nonzero idempotent of R can be expressed as a sum of orthogonal primitive idempotents of R; (2) for a regular ring R such that $I(R)$ is a finite additive set, if the multiplicative group of all units of R is abelian (resp. cyclic), then R is a commutative ring (resp. R is a finite direct product of finite fields).

1. Introduction and basic definitions

Throughout this paper, let R be an associative ring with identity 1. The Jacobson radical of R is denoted by $J(R)$. We use $I(R)$ for the set of all nonunit idempotents of R, while we let $M(R)$ be the set of all primitive idempotents and 0 of R. We use $Z(R)$ and $\mathrm{char}(R)$ to denote the center of R and the characteristic of R, respectively. A nonempty subset of a ring R is called *multiplicative* if it is closed under

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multiplication. Recall that two idempotents $e, f \in R$ are said to be *orthogonal* if $ef = fe = 0$. Also recall that a nonzero idempotent $e \in R$ is said to be *primitive* if it can not be written as a sum of two nonzero orthogonal idempotents, or equivalently, eR (resp. Re) is indecomposable as a right (resp. left) R-module. Recall that R is said to have a complete set of primitive idempotents if there exists a finite set of mutually orthogonal primitive idempotents whose sum is 1.

In [1], Dolžan has shown that a finite ring R with $M(R)$ multiplicative is a product of local rings. In [2], Grover et al. have extended Dolžan's result as follows: if R is a ring with a complete set of primitive idempotents, then $M(R)$ is multiplicative if and only if R is a finite direct product of connected rings. On the other hand, in [5], it was shown that in case that R is a direct product of countably many (not finite) connected rings $M(R)$ could not be multiplicative.

We say that $I(R)$ is *additive* if for all $e, f \in I(R)$ $(e \neq f)$, $e+f \in I(R)$ (equivalently, $ef = -fe$). For example, if R is a Boolean ring, then $I(R)$ is additive. Also $M(R)$ is said to be *additive in* $I(R)$ if for all $e, f \in M(R)$ $(e \neq f)$, $e + f \in I(R)$. For example, if R is a Boolean ring or a direct product of local rings, then $M(R)$ is additive in $I(R)$. Note that if $I(R)$ is additive, then $M(R)$ is additive in $I(R)$, but the converse is not true by considering a finite direct product of infinite fields. We also note that $I(R)$ is commuting if and only if $I(R)$ is multiplicative if and only if $I(R) \subseteq Z(R)$. By [5, Lemma 1] if $I(R)$ is additive, then $I(R) \subseteq Z(R)$. But the converse may not be true (e.g., $\mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$). In [5], it was shown that $I(R)$ is additive if and only if $I(R)$ is commuting and $\mathrm{char}(R) = 2$; $M(R)$ is additive in $I(R)$ if and only if $M(R)$ is the set of primitive pairwise orthogonal idempotents.

We call a nonzero idempotent e in a ring R $fully\ basic$ if e can be expressed as a sum of mutually orthogonal primitive idempotents in R, and we call a ring R a $fully\ basic\ ring$ if all idempotents are fully basic. For example, a finite direct product of local rings and $T_2(\mathbb{Z}_2)$, the ring of all upper triangular 2×2 matrices over $\mathbb{Z}_2$, are fully basic rings. Note that in a fully basic ring R (e.g., $T_2(\mathbb{Z}_2)$), $I(R)$ may not be multiplicative.

In this paper, we will investigate a ring R such that $I(R) \neq \{0\}$ is a finite additive set. In Section 2, we will show that if $I(R)$ is a finite additive set of a ring R, then there exists at least one primitive idempotent, and we will also show that $I(R)$ is a finite additive set of a ring R if and only if $M(R) \setminus \{0\}$ is a complete set of minimal central idempotents, the characteristic of R (denoted by $\mathrm{char}(R)$) is 2 and R is fully basic.

Recall that a ring R is *von-Neumann regular* (simply *regular*) (resp. *unit-regular*) provided that for any $a \in R$ there exists an element $r \in R$ (resp. a unit $u \in R$) such that $a = ara$ (resp. $a = aua$). A ring R is *strongly regular* provided that for any $a \in R$ there exists some element $r \in R$ such that $a = ra^2$. Also a ring R is *abelian* provided all idempotents in R are central. In section 3, we will show that for a regular ring R such that $I(R)$ is a finite additive set, if G, the group of all units of R, is an abelian (resp. a cyclic) group, then R is a commutative ring (resp. R is a finite direct product of finite fields).

2. Some properties of a ring with a finite additive set of idempotents

Throughout this section, we assume that $I(R) \neq \{0\}$ for any ring R. Let $\preceq$ denote the usual relation on $I(R)$, that is, $e \preceq f$ (or $f \succeq e$) means that $ef = fe = e$ (refer [1]). In particular, $e \prec f$ (or $f \succ e$) means that $e \preceq f$ and $e \neq f$. A nonzero idempotent e is called *minimal* if there is no idempotent strictly between 0 and e according to the partial ordering $\preceq$. Note that the minimal idempotents in this sense are precisely the primitive idempotents of R.

Lemma 2.1. *Let R be a ring. Then we have the following:*
(1) [5, Theorem 2.5] $I(R)$ is additive if and only if $I(R)$ is commuting and char$(R) = 2$.
(2) [5, Corollary 2.6] $M(R)$ is additive in $I(R)$ if and only if $M(R)$ is the set of mutually primitive orthogonal idempotents.

Lemma 2.2. *Let R be a ring such that $I(R)$ is an additive set and let $0 \neq e \in I(R)$. If $ce = 0$ for all $c \in I(R)$ $(c \neq e)$, then e is primitive.*

Proof. Assume that e is not primitive. Then $e = a+b$ for some nonzero orthogonal idempotents a, b of R. Since e is not primitive, $a, b \neq e$. By assumption, $0 = ae = a + ab$, and $0 = be = ba + b$ and so $a = b = 0$ since a, b are orthogonal, a contradiction. Hence e is primitive. □

Lemma 2.3. *Let R be a ring. If $I(R)$ is a finite additive set in R, then $M(R) \neq \{0\}$.*

Proof. Note that if $I(R)$ is orthogonal (i.e., $ab = ba = 0$ for all $a, b \in I(R)$), then each nonzero $e \in I(R)$ is primitive. Indeed, assume that $e \in I(R)$ is not primitive. Then $e = a + b$ for some nonzero orthogonal idempotents a, b of R. Clearly, $a \neq b$. If $a \neq e$ (resp. $b \neq e$), then $0 = ea = a$ (resp. $0 = eb = b$), a contradiction. Hence each $e \in I(R)$ is primitive. Suppose that $I(R)$ is not orthogonal. Then there exist $e, f \in I(R)$ $(e \neq f)$ such that $ef \neq 0$. Thus $e \succeq ef$. If ef is primitive, we are done. If ef is not primitive, there exists a nonzero $e_1 \in I(R)$ such that $e_1(ef) \neq 0$ by Lemma 2.2. Thus $ef \succ e_1(ef)$. Continuing this procedure then, starting now with e_1, we arrive at a strictly descending relation

$$ef \succ e_1(ef) \succ e_2e_1(ef) \succ \cdots .$$

Since $I(R)$ is finite, this relation terminates with some nonzero $e_t \cdots e_1(ef) \in I(R)$, and $e_t \cdots e_1(ef)$ must then be primitive. Hence $M(R) \neq \{0\}$. □

THEOREM 2.4. *Let R be a ring such that $I(R)$ is a finite additive set. Then we have the following:*

(1) *R is fully basic.*
(2) *If $e = e_1 + \cdots + e_s = f_1 + \cdots + f_t$ for any nonzero $e \in I(R)$ where all e_i's (resp. f_j's) are mutually orthogonal primitive idempotents of R, then $s = t$ and f_j can be renumbered so that $e_i = f_i$.*

Proof. (1) Let $0 \neq e \in I(R)$ be arbitrary. We have $M(R) \neq \{0\}$ by Lemma 2.3. If e is primitive, then we are done. Suppose that e is not primitive. Then by the proof given in Lemma 2.3, there exists a nonzero $f_1 \in I(R)$ such that f_1e $(= ef_1)$ is primitive, and so $e = ef_1 + (e - ef_1)$, which is a sum of orthogonal idempotents of R. Note that $e \succ (e - ef_1)$. If $e - ef_1$ is primitive, then we are done. Suppose that $e - ef_1$ is not primitive. By the similar argument, there exists a nonzero $f_2 \in I(R)$ such that$(e - ef_1)f_2$ is primitive. Thus $e - ef_1 = (e - ef_1)f_2 + ((e - ef_1) - (e - ef_1)f_2))$, which is also a sum of orthogonal idempotents of R. Also note that $e - ef_1 \succ ((e - ef_1) - (e - ef_1)f_2))$. Continuing in this procedure, we get a strictly descending sequence of relations

$$a_0 \succ a_1 \succ a_2 \succ \cdots$$

where $a_0 = e$, $a_{n+1} = a_n - a_nf_{n+1}$ with $a_nf_{n+1} \in M(R)$ for some nonzero idempotent f_{n+1} of R and $a_n \neq 0$ for all $n = 1, 2, \ldots$. Next, we will show that all f_n are distinct. To show this, we will proceed it by induction on n. If $n = 2$, then clearly, $f_1 \neq f_2$. Assume that this holds for n, i.e., $f_i \neq f_j$ for all distinct i, j $(1 \leq i, j \leq n)$. For $n+1$, it is enough to show that $f_{n+1} \neq f_i$ for all $i = 1, \ldots, n$. Assume that $f_{n+1} = f_i$ for some i $(1 \leq i \leq n)$. Then $a_n f_{n+1} = (a_{n-1} - a_{n-1} f_n) f_{n+1} = (a_{n-1} - a_{n-1} f_n) f_i = a_{n-1} f_i = (a_{n-2} - a_{n-2} f_{n-1}) f_i = a_{n-2} f_i = \cdots = a_i f_i = 0$, which is a contradiction to $a_n f_{n+1} \in M(R)$. Hence $f_{n+1} \neq f_i$ for all $i = 1, \ldots, n$. Since $I(R)$ is finite and all f_n are distinct, the above sequence must terminate, and so a_n is a primitive idempotent of R. Hence $e = a_0 f_1 + a_1 f_2 + \cdots + a_{n-1} f_n + a_n$, which is a sum of orthogonal primitive idempotents in R.

(2) We can let $s \leq t$ without loss of generality. Since $e = e_1 + \cdots + e_s = f_1 + \cdots + f_t$ where all e_i's (resp. f_j's) are mutually orthogonal primitive idempotents of R, $e_1 = e_1 e = e_1 f_1 + \cdots + e_1 f_t$. Since e_1 is a primitive idempotent of R, $e_1 = e_1 f_1$ and $e_1 f_2 = \cdots = e_1 f_t = 0$ by renumbering f_j. Also, we have $f_1 = e f_1 = e_1 f_1 + \cdots + e_s f_t$. Since f_1 is a primitive idempotent of R and $e_1 f_1 \neq 0$, $f_1 = e_1 f_1 = e_1$. Thus $e_2 + \cdots + e_s = f_2 + \cdots + f_t$. Continuing in this way, we also have that $e_2 = f_2, \ldots, e_s = f_s$ by renumbering f_j. Then $f_{s+1} + \cdots + f_t = 0$, which implies that $f_{s+1} = \cdots = f_t = 0$. Hence we have the result. $\square$

Let R be a ring such that $I(R)$ is a finite additive set. Then any nonzero $e \in I(R)$ can be expressed uniquely as a sum of a finite number of orthogonal primitive idempotents in R by Theorem 2.4. Here the unique number is called the *length* of e and is denoted by $\ell(e)$.

Lemma 2.5. *Let R be a ring such that $I(R)$ is a finite additive set and let $e = e_1 + e_2 + \cdots + e_s, f = f_1 + f_2 + \cdots + f_t \in I(R)$ with $\ell(e) = s, \ell(f) = t$ where all e_i's (resp. f_j's) are mutually orthogonal primitive idempotents of R. If $ef = 0$, then $e_i f_j = 0$ for all i, j.*

Proof. First, we observe that if $e_i f_j, e_k f_\ell \neq 0$ where $i \neq k$ or $j \neq \ell$, then $e_i f_j \neq e_k f_\ell$. Indeed, without loss of generality, we can let $i \neq k$. If $e_i f_j = e_k f_\ell$, then $e_i f_j = e_i(e_k f_\ell) = (e_i e_k) f_\ell = 0$, a contradiction. Note that $e_i f_j = \sum e_k f_\ell$ for all i, j $(i \neq k$ or $j \neq \ell)$. Thus $e_i f_j = e_i(e_i f_j) f_j = e_i(\sum e_k f_\ell) f_j = (\sum_{j \neq \ell} e_i f_\ell) f_j = 0$. $\square$

Lemma 2.6. *Let R be a ring. If $I(R)$ is a finite additive set, then $M(R) \setminus \{0\}$ is a complete set of primitive central idempotents.*

Proof. By Lemma 2.3, $M(R) \neq \{0\}$. Since $I(R)$ is finite, we can let $M(R) \setminus \{0\} = \{e_1, e_2, \cdots, e_r\}$. Since $I(R)$ is additive, all idempotents are central by Lemma 2.1. Since $I(R)$ is additive, $M(R)$ is clearly additive in $I(R)$. Hence $M(R) \setminus \{0\}$ is orthogonal Lemma 2.2. Thus $\{e_1, e_2, \ldots, e_r\}$ is the set of primitive central idempotents of R. To prove that $\{e_1, e_2, \ldots, e_r\}$ is a complete set of primitive central idempotents, it remains to show that $1 = e_1 + e_2 + \cdots + e_r$. Consider $e = e_1 + e_2 + \cdots + e_{r-1} \in I(R)$. Note that $e \neq 0, 1$ and $1 = e + (1 - e)$, which is a sum of orthogonal idempotents of R. By Theorem 2.4, there exist mutually orthogonal primitive idempotents $f_1, f_2, \ldots, f_s$ of R such that $1 - e = f_1 + f_2 + \cdots + f_s$. Assume that $s \geq 2$. Let $T = \{e_1, \ldots, e_{r-1}, f_1, \ldots, f_s\}$. Then since $e(1 - e) = 0$, $e_i f_j = 0$ for all i, j by Lemma 2.5. Thus T is orthogonal with $|T| = r - 1 + s > r = |M(R) \setminus \{0\}|$. Since $T \subseteq M(R) \setminus \{0\}$, we arrive at a contradiction. Hence $s = 1$, and then $f_1 = e_r$. Therefore, we have $1 = e_1 + e_2 + \cdots + e_r$. □

THEOREM 2.7. *Let R be a ring. Then $I(R)$ is a finite additive set if and only if $M(R) \setminus \{0\}$ is a complete set of primitive central idempotents, char(R) $= 2$ and R is fully basic.*

Proof. (⇒) It follows from Lemma 2.1, 2.6 and Theorem 2.4.

(⇐) Suppose that $M(R) \setminus \{0\}$ is a complete set of primitive central idempotents, char$(R) = 2$ and R is fully basic. Since $M(R) \setminus \{0\}$ is finite and R is fully basic, $I(R)$ is clearly finite. To show that $I(R)$ is additive, let e, f be arbitrary nonzero distinct idempotents of R. Since R is fully basic, then $e = e_1 + e_2 + \cdots + e_r$ and $f = f_1 + f_2 + \cdots + f_s$ where all e_i's (resp. f_j's) are mutually orthogonal primitive idempotents of R. Since char$(R) = 2$, we can assume that all e_i, f_j are distinct. Since $M(R) \setminus \{0\}$ is orthogonal, $(e + f)^2 = e + f$, and so $I(R)$ is additive. □

COROLLARY 2.8. *Let R be a ring. If $I(R)$ is a finite additive set, then R is a finite direct product of indecomposable rings and $|I(R) \cup \{1\}| = 2^r$ where $|M(R) \setminus \{0\}| = r$.*

Proof. Let $M(R) \setminus \{0\} = \{e_1, e_2, \ldots, e_r\}$. By Lemma 2.6, $M(R) \setminus \{0\}$ is a complete set of primitive central idempotents. Since $1 = e_1 + e_2 + \cdots + e_r$, for all $a \in R$, $a = e_1 a + e_2 a + \cdots + e_r a$, which is a sum of mutually orthogonal elements of R, and so $R = e_1 R \oplus e_2 R \oplus \cdots \oplus e_r R$, which is a finite direct product of indecomposable rings. Since each $e_i \in M(R) \setminus \{0\}$ is a primitive idempotent, $|I(e_i R)| = 2$, and so $|I(R)| = 2^r$ where $|M(R) \setminus \{0\}| = r$. □

Remark 1. *Note that if R is a ring such that I(R) is additive, then* $I(R) \cup \{1\}$ *forms a Boolean subring of R. In particular, if I(R) is a finite additive set, then* $I(R) \cup \{1\} \simeq \underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{r-summands}$ $(r = |M(R) \setminus \{0\}|)$.

Corollary 2.9. *Every finite Boolean ring R is isomorphic to* $\underbrace{\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2}_{r-summands}$ *where* $r = |M(R) \setminus \{0\}|$.

Proof. It follows from Remark 1. □

3. A von-Neumann regular ring with a finite additive set of idempotents

Let R be a ring, $X(R)$ (simply, denoted by X) the set of all nonzero, nonunits of R, $G(R)$ (simply, denoted by G) the group of all units of R. In this section, we will consider a group action of G on X given by $((g,x) \longrightarrow gx)$ from $G \times X$ to X, called the regular action. For each $x \in X$, we define the *orbit* of x by $o(x) = \{gx : \forall g \in G\}$ under the regular action of G on X.

The following lemma was shown in [4, Lemma 2.3].

Lemma 3.1. *Lemma 3.1 Let R be a ring such that G acts on X by the regular action. Then R is unit-regular if and only if every orbit under the regular action is* $o(e)$ *for some idempotent* $e \in X$.

Remark 2. *Let R be a ring such that I(R) is a finite additive set. Then we note that (1) R is regular if and only if R is unit-regular if and only if R is strongly regular if and only if R is abelian regular; (2) In a regular ring R, there are a finite number of orbits under the regular action of G on X.*

Theorem 3.2. *Let R be an abelian regular ring. If G is an abelian group, then R is a commutative ring.*

Proof. First, let $x \in X$ and $g \in G$ be arbitrary. Since R is abelian regular, R is unit-regular. Thus there exists an element $u \in G$ such that $x = xux$, and so $ux, xu \in I(R)$. Since R is abelian, xu and ux are central. Since G is abelian, $(gx)u = g(xu) = (xu)g = x(ug) = x(gu) = (xg)u$, and so $gx = xg$. Next, let $x, y \in X$ be arbitrary. If $x \in I(R)$, then $xy = yx$. If $x \notin I(R)$, then $vx, xv \in I(R)$ for some

$v \in G$. Then $v(xy) = (vx)y = y(vx) = (yv)x = (vy)x = v(yx)$ by the above argument, and so $xy = yx$. Consequently, R is a commutative ring. □

COROLLARY 3.3. *Let R be a regular ring such that $I(R)$ is a finite additive set. If G is abelian, then R is a commutative ring.*

Proof. It follows from Remark 2 and Theorem 3.2. □

THEOREM 3.4. *Let R be an abelian regular ring having a complete set of primitive idempotents. If G is cyclic, then R is a finite direct product of finite fields.*

Proof. Let $S = \{e_1, e_2, \ldots, e_r\}$ be a complete set of central primitive idempotents in R. Then $R = e_1R \times e_2R \times \cdots \times e_rR$, a finite product of local rings. Note that since the Jacobson radical of R is zero, each e_iR is a division ring. Since G is abelian, R is a commutative ring by Theorem 3.2, and then each e_iR is a field. Since G is cyclic, each $G(e_iR)$ is also cyclic, and so e_iR is finite by [6, Exercise 12, p. 426]. Hence R is a finite direct product of finite fields. □

COROLLARY 3.5. *Let R be a regular ring such that $I(R)$ is a finite additive set. If G is cyclic, then R is a finite direct product of finite fields of characteristic 2 with distinct orders.*

Proof. It follows from Remark 2 and Theorem 3.4. □

THEOREM 3.6. *Let R be an abelian regular ring with a complete set of primitive idempotents. If G is finite, then R is finite.*

Proof. Let $x \in X$ be arbitrary. Then $x = ge$ for some $g \in G$ and some $e \in I(R)$ by Lemma 3.1. Let $\{e_1, e_2, \ldots, e_r\}$ be a complete set of primitive idempotents of R. Since $1 = e_1 + e_2 + \cdots + e_r$, $x = ge = \sum_{ee_i \neq 0} g(ee_i)$. Since G is finite, $o(ee_i)$ is finite for all $ee_i \neq 0$. Hence X is finite, and then R is finite by [3, Theorem 2.2]. □

COROLLARY 3.7. *Corollary 3.7 Let R be a regular ring such that $I(R)$ is a finite additive set. Then we have the following:*

(1) *If G is finite, then R is finite.*
(2) *$G = \{1\}$ if and only if R is a finite Boolean ring.*

Proof. (1) It follows from Remark 2 and Theorem 3.6.
(2) By (1), if $G = \{1\}$, then R is finite by (1). Since R is a regular ring such that $I(R)$ is a finite additive set, R is unit-regular by Remark 2.

Let $x \in X$ be arbitrary. Then $x = ge$ for some $g \in G$ and $e \in I(R)$ by Lemma 3.1. Since $G = \{1\}$, $x = e$, and so $X = I(R) \setminus \{0\}$. Hence $R = I(R) \cup \{1\}$ is a Boolean ring. The converse follows from Corollary 2.11. $\square$

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