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SOME NOTES ON STRONG LAW OF LARGE NUMBERS FOR BANACH SPACE VALUED FUZZY RANDOM VARIABLES

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Abstract. In this paper, we establish two types of strong law of large numbers for fuzzy random variables taking values on the space of normal and upper-semicontinuous fuzzy sets with compact support in a separable Banach space. The first result is SLLN for strong-compactly uniformly integrable fuzzy random variables, and the other is the case of that the averages of its expectations converges.

1. Introduction

In the recent years, there have been increasing interests in limit theorems for random sets and fuzzy random variables because of its usefulness in several applied fields. Among others, several variants of strong law of large numbers(SLLN) for random sets and fuzzy random variables have been developed by many researchers. SLLN for random sets were obtained by Artstein and Hansen [1], Artstein and Vitale [2], Puri and Ralescu [22], Taylor and Inoue [24-26], and so on.

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The concept of a fuzzy random variables as a generalization of a random set was introduced by Puri and Ralescu [23] in order to handle inexact data due to the subjectivity and imprecision of human knowledge.

Since then, several people have studied strong laws of large numbers for independent fuzzy random variables based on the limit theorems for random sets. For example, Klement et al. [17] proved some limit theorems which include a SLLN for i.i.d. fuzzy random variables and Inoue [11] obtained a SLLN for independent and tight case. In their works, the L_1-metric on the space of fuzzy sets was used. As results of SLLN using the supremum metric on the space of fuzzy sets, Colubi et al [3] obtained SLLN by the approximation method and Molchanov [20] gave a short proof of SLLN for i.i.d. case. Besides that, a rich variety of SLLN for fuzzy random variables can be found in [4, 7, 12, 15, 21, 27].

As the latest results, SLLN for weighted sum of fuzzy random variables were established by Guan and Li [10], Joo et al. [14] and SLLN for arrays of row-wise independent case was given by Fu and Zhang [8].

The purpose of this paper is to obtain some results on SLLN for fuzzy random variables. In fact, this paper was motivated by Li and Ogura [18] which presented SLLN for independent and compactly uniformly integrable fuzzy random variables by assuming additional restrictive condition that the sequence of averages of its expectations converges to some fuzzy sets. In this paper, we first prove that the additional restrictive condition in Li and Ogura [18] can be deleted in the case of strong-compactly uniformly integrable case. And then, we give SLLN for integrably bounded fuzzy random variables only by assuming that the sequence of averages of its expectations is convergent.

2. Preliminaries

Let Y be a real separable Banach space with norm $|\cdot|$ and let $\mathbf{K}(Y)$ denote the family of all non-empty compact subsets of Y. Then the space $\mathbf{K}(Y)$ is metrizable by the Hausdorff metric h defined by

$$h(A,B)=\max\{\sup_{a\in A}\inf_{b\in B}|a-b|,\sup_{b\in B}\inf_{a\in A}|a-b|\}.$$

A norm of $A\in\mathbf{K}(Y)$ is defined by

$$\|A\|=h(A,\{0\})=\sup_{a\in A}|a|.$$

It is well-known that $\mathbf{K}(Y)$ is complete and separable with respect to the Hausdorff metric h (See Debreu [6]). The addition and scalar multiplication on $\mathbf{K}(Y)$ are defined as usual:

$$\begin{aligned} A \oplus B &= \{a + b : a \in A, b \in B\} \\ \lambda A &= \{\lambda a : a \in A\} \end{aligned}$$

for $A, B \in \mathbf{K}(Y)$ and $\lambda \in R$.

The convex hull and closed convex hull of $A \subset Y$ are denoted by $co(A)$ and $\overline{co}(A)$, respectively. Then it is well-known that $co(A)$ may not be an element of $\mathbf{K}(Y)$ even though $A \in \mathbf{K}(Y)$ but $\overline{co}(A) \in \mathbf{K}(Y)$ if $A \in \mathbf{K}(Y)$.

Let $\mathbf{F}(Y)$ denote the family of all fuzzy sets $u : Y \to [0, 1]$ with the following properties;

(1) u is normal, i.e., there exists $x \in Y$ such that $u(x) = 1$;

(2) u is upper semicontinuous;

(3) $\operatorname{supp} u = \overline{\{x \in Y : u(x) > 0\}}$ is compact, where $\overline{A}$ denotes the closure of a set $A \subset Y$.

For a fuzzy set u in Y, the α-level set of u is defined by

$$L_\alpha u = \begin{cases} \{x : u(x) \geq \alpha\}, & \text{if } \ 0 < \alpha \leq 1 \\ \operatorname{supp} u, & \text{if } \ \alpha = 0. \end{cases}$$

Then, it follows immediately that

$$u \in \mathbf{F}(Y) \text{ if and only if } \ L_\alpha u \in \mathbf{K}(Y) \ \text{ for each } \alpha \in [0, 1].$$

The linear structure on $\mathbf{F}(Y)$ is defined as usual;

$$(u \oplus v)(z) = \sup_{x+y=z} \min(u(x), v(y)),$$

$$(\lambda u)(z) = \begin{cases} u(z/\lambda) & , \quad \lambda \neq 0 \\ \tilde{0} & , \quad \lambda = 0, \end{cases}$$

for $u, v \in \mathbf{F}(Y)$ and $\lambda \in R$, where $\tilde{0} = I_{\{0\}}$ denotes the indicator function of $\{0\}$. Then it is known that for each $\alpha \in [0, 1]$,

$$L_\alpha(u \oplus v) = L_\alpha u \oplus L_\alpha v$$

and

$$L_\alpha(\lambda u) = \lambda L_\alpha u.$$

Lemma 2.1. *For $u \in \mathbf{F}(Y)$, we define*

$$F_u : [0,1] \longrightarrow (\mathbf{K}(Y), h), F_u(\alpha) \;=\; L_\alpha u.$$

Then the followings hold;

(1) *F_u is non-increasing, i.e., $\alpha \leq \beta$ implies $F_u(\alpha) \supset F_u(\beta)$,*

(2) *F_u is left continuous on $(0,1]$,*

(3) *F_u has right-limits on $[0,1)$and F_u is right-continuous at 0.*

Conversely, if $G : [0,1] \to \mathbf{K}(Y)$ is a function satisfying the above conditions $(1)-(3)$, then there exists a unique $v \in \mathbf{F}(Y)$ such that $G(\alpha) = L_\alpha v$ for all $\alpha \in [0,1]$.

Proof. See Lemma 2.2 of Joo and Kim [13]. □

We denote $\overline{\{x \in Y : u(x) > \alpha\}}$ by $L_{\alpha^+}u$. Then the right limit of F_u at α is $L_{\alpha^+}u$.

We can easily show that for $u \in \mathbf{F}(Y)$, the function $G : [0,1] \to \mathbf{K}(Y)$ defined by $G(\alpha) = \overline{co}(L_\alpha u)$ satisfy the conditions $(1) - (3)$ in Lemma 2.1. Thus there exists a unique $v \in \mathbf{F}(Y)$ such that $G(\alpha) = L_\alpha v$ for all $\alpha \in [0,1]$.

This fuzzy set v is called the closed convex hull of u and denoted by $\overline{co}(u)$. So, $\overline{co}(u) \in \mathbf{F}(Y)$ and $L_\alpha \overline{co}(u) = \overline{co}(L_\alpha u)$ for each $\alpha \in [0,1]$.

Now, we define the supremum metric d_∞ on $F(Y)$ by

$$d_\infty(u,v) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha u, L_\alpha v).$$

Also, the norm of u is defined as

$$\|u\| = d_\infty(u, \tilde{0}) = \sup_{x \in L_0 u} |x|.$$

Then it is well-known that $\mathbf{F}(Y)$ is complete but is not separable with respect to d_∞(see Klement et al. [17]).

3. Main Results

Throughout this paper, let $(\Omega, \mathcal{A}, P)$ be a probability space. A set-valued function $X : \Omega \to \mathbf{K}(Y)$ is called measurable if for each closed subset B of Y,

$$X^{-1}(B) = \{\omega : X(\omega) \cap B \neq \emptyset\} \in \mathcal{A}.$$

It is well-known that the measurability of X is equivalent to the measurability of X considered as a map from Ω to the metric space $(\mathbf{K}(Y), h)$. A set-valued function $X : \Omega \to \mathbf{K}(Y)$ is called a random set if it is measurable.

A random set X is called integrably bounded if

$$E\|X\| < \infty.$$

The expectation of integrably bounded random set X is defined by

$$E(X) = \{E(\xi) : \xi \in L(\Omega, Y) \,\text{and}\, \xi(\omega) \in X(\omega)\, a.s.\},$$

where $L(\Omega, Y)$ denotes the class of all Y-valued random variables ξ such that $E|\xi| < \infty$.

A fuzzy set valued function $\tilde{X} : \Omega \to \mathbf{F}(Y)$ is called a fuzzy random variable (or fuzzy random set) if for each $\alpha \in [0,1], L_\alpha \tilde{X}$ is a random set. If $\tilde{X} : \Omega \to (\mathbf{F}(Y), d_\infty)$ is measurable, then $\tilde{X}$ is a fuzzy random variable. But the converse is not true.(see Colubi et al. [5] or Kim [16])

A fuzzy random variable $\tilde{X}$ is called integrably bounded if $E\|\tilde{X}\| < \infty$. The expectation of integrably bounded fuzzy random variable $\tilde{X}$ is a fuzzy subset $E(\tilde{X})$ of Y defined by

$$E(\tilde{X})(x) = \sup\{\alpha \in [0,1] : x \in E(L_\alpha \tilde{X})\}.$$

For details for expectations of random sets and fuzzy random variables, the readers refer to Li et al [19].

We introduce the concepts of tightness and compact uniform integrability for a sequence of random sets and fuzzy random variables.

Definition 3.1. *Let $\{X_n\}$ be a sequence of random sets.*

(1) *$\{X_n\}$ is said to be tight if for each $\epsilon > 0$, there exists a compact subset $\mathcal{K}$ of $(\mathbf{K}(Y), h)$ such that*

$$P(X_n \notin \mathcal{K}) < \epsilon \quad \textit{for all } n.$$

(2) *$\{X_n\}$ is said to be compactly uniformly integrable(CUI) if for each $\epsilon > 0$, there exists a compact subset $\mathcal{K}$ of $(\mathbf{K}(Y), h))$ such that*

$$\int_{\{X_n \notin \mathcal{K}\}} \|X_n\| \, dP \; < \; \epsilon \; \textit{for all } n.$$

Definition 3.2. *Let $\{\tilde{X}_n\}$ be a sequence of fuzzy random variables.*

(1) *$\{\tilde{X}_n\}$ is said to be level-wise independent if for each $\alpha \in [0,1]$, the sequence of random sets $\{L_\alpha \tilde{X}_n\}$ is independent.*

(2) *$\{\tilde{X}_n\}$ is said to be independent if the sequence of σ-fields $\{\sigma(\tilde{X}_n)\}$ is independent, where $\sigma(\tilde{X})$ is the smallest σ-field which $L_\alpha \tilde{X}$ is measurable for all $\alpha \in [0,1]$.*

(3) *$\{\tilde{X}_n\}$ is said to be tight if for each $\epsilon > 0$, there exists a compact subset $\mathcal{K}$ of $(\mathbf{K}(Y), h)$ such that*

$$P(L_\alpha \tilde{X}_n \notin \mathcal{K}) < \epsilon \ \text{ for all } \ n \text{ and all } \ \alpha \in [0,1].$$

(4) *$\{\tilde{X}_n\}$ is said to be strongly tight if for each $\epsilon > 0$, there exists a compact subset K of $(\mathbf{F}(Y), d_\infty)$ such that*

$$P(\tilde{X}_n \notin K) < \epsilon \text{ for all } \ n.$$

(5) *$\{\tilde{X}_n\}$ is said to be compactly uniformly integrable (CUI) if for each $\epsilon > 0$ there exists a compact subset $\mathcal{K}$ of $(\mathbf{K}(Y), h)$ such that*

$$\int_{\{L_\alpha \tilde{X}_n \notin \mathcal{K}\}} \|L_\alpha \tilde{X}_n\| \, dP \ < \ \epsilon \text{ for all } n \ \text{ and all } \ \alpha \in [0,1].$$

(6) *$\{\tilde{X}_n\}$ is said to be strong-compactly uniformly integrable (SCUI) if for each $\epsilon > 0$ there exists a compact subset K of $(\mathbf{F}(Y), d_\infty)$ such that*

$$\int_{\{\tilde{X}_n \notin \mathcal{K}\}} \|\tilde{X}_n\| \ dP < \epsilon \text{ for all } n.$$

It is trivial that strong-compactly uniform integrability (resp. strong tightness) implies compactly uniform integrability (resp. tightness). The following example shows that the converse is not true even though Y is finite dimensional.

Example. Let $Y = R$ be the real line. For $0 < \lambda < 1$, let

$$u_\lambda(x) = \begin{cases} 1, & if \ \ x = 0, \\ \lambda, & if \ \ 0 < |x| \le 1 \\ 0, & elsewhere. \end{cases}$$

Then

$$L_\alpha u_\lambda = \begin{cases} \{0\}, & if \ \ \lambda < \alpha \le 1 \\ \{x : |x| \le 1\}, & if \, 0 \le \alpha \le \lambda, \end{cases}$$

and so $d_\infty(u_\lambda, u_\delta) = 1$ for $\lambda \neq \delta$.

Now let $\{\lambda_n\}$ be a sequence of distinct elements in $[0,1]$ and $\tilde{X}_n$ be a fuzzy random variable with $P(\tilde{X}_n = u_{\lambda_n}) = 1$.

Suppose that $0 < \epsilon < 1$ and there is a compact subset K of $(\mathbf{F}(R), d_\infty)$ such that

$$\int_{\{\tilde{X}_n \notin K\}} \|\tilde{X}_n\| \, dP < \epsilon \text{ for all } n.$$

Then since

$$P\{\tilde{X}_n \notin K\} = \int_{\{\tilde{X}_n \notin K\}} \|\tilde{X}_n\| \, dP < \epsilon,$$

K necessarily contains λ_n for all n. But this is impossible because a sequence $\{u_{\lambda_n}\}$ does not have any convergent subsequence. Thus, $\{\tilde{X}_n\}$ cannot be strong-compactly uniformly integrable.

But if we take $\mathcal{K} = \{A \in \mathbf{K}(R) : \|A\| \leq 1\}$, then $\mathcal{K}$ is a compact subset of $(\mathbf{K}(R), h)$ and

$$\int_{\{L_\alpha \tilde{X}_n \notin \mathcal{K}\}} \|L_\alpha X_n\| \, dP \;=\; 0 \text{ for all } n \text{ and all } \alpha \in [0,1],$$

which implies compactly uniform integrability of $\{\tilde{X}_n\}$.

The following theorem was obtained by Li and Ogura [18].

Theorem 3.1. (*Li and Ogura [18]*) *Let $\{\tilde{X}_n\}$ be a sequence of level-wise independent and CUI fuzzy random variables. If*

$$\sum_{n=1}^{\infty} \frac{1}{n^p} E\|\tilde{X}_n\|^p < \infty \textit{ for some } 1 \leq p \leq 2,$$

and $\{\frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{X}_i)\}$ is convergent with respect to d_∞, then

$$\lim_{n\to\infty} d_\infty\Big(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, \frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{X}_i)\Big) \;=\; 0 \; a.s..$$

If we assume strong-compactly uniform integrablilty, then we can obtain SLLN without the additional restrictive assumption that $\{\frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{X}_i)\}$ is convergent.

Theorem 3.2. *Let $\{\tilde{X}_n\}$ be a sequence of SCUI fuzzy random variables. Then*

$$\lim_{n\to\infty} d_\infty\Big(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, \frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{X}_i)\Big) \;=\; 0 \, a.s.$$

if and only if for each $\alpha \in [0,1]$,

$$\lim_{n\to\infty} h(\frac{1}{n} \oplus_{i=1}^{n} L_\alpha \tilde{X}_i, \frac{1}{n} \oplus_{i=1}^{n} \overline{co}(EL_\alpha \tilde{X}_i)) \to 0 \ \ a.s.$$

and

$$\lim_{n\to\infty} h(\frac{1}{n} \oplus_{i=1}^{n} L_{\alpha^+} \tilde{X}_i, \frac{1}{n} \oplus_{i=1}^{n} \overline{co}(EL_{\alpha^+} \tilde{X}_i)) \to 0 \ \ a.s..$$

To prove the above theorem, we need some lemmas.

Lemma 3.3. (*Greco and Moschen* [9]) *Let K be subset of* $(\mathbf{F}(Y), d_\infty)$. *Then K is relatively compact if and only if*

(1) *K is compact-supported, i.e., there is a compact subset A of Y such that $L_0 u \subset A$ for all $u \in K$.*

(2) *For each* $\alpha \in (0,1]$, $\lim_{\delta\to 0} \sup_{u\in K} h(L_\alpha u, L_{\alpha-\delta} u) = 0$.

(3) *For each* $\alpha \in [0,1)$, $\lim_{\delta\to 0} \sup_{u\in K} h(L_{\alpha^+} u, L_{\alpha+\delta} u) = 0$.

Lemma 3.4. *Let K be a relatively compact subset of* $(\mathbf{F}(Y), d_\infty)$. *Then* $\{\overline{co}(u) : u \in K\}$ *is also relatively compact in* $(\mathbf{F}(Y), d_\infty)$.

Proof. It follows from the facts that if $L_0 u \subset A$, then $L_0 \overline{co}(u) = \overline{co}(L_0 u) \subset \overline{co}(A)$ and

$$h(L_\alpha \overline{co}(u), L_\beta \overline{co}(v)) = h(\overline{co} L_\alpha u, \overline{co} L_\beta v) \leq h(L_\alpha u, L_\beta v).$$

□

Recall that we can define the concept of convexity on $\mathbf{F}(Y)$ as in the case of a vector space even though $\mathbf{F}(Y)$ is not a vector space. That is, $K \subset \mathbf{F}(Y)$ is said to be convex if $\lambda u \oplus (1-\lambda) v \in K$ whenever $u, v \in K$ and $0 \leq \lambda \leq 1$. Also, the convex hull *co*(K) of K is defined to be the intersection of all convex sets that contains K. Then we can easily show that *co*(K) is equal to the family of consisting of all fuzzy sets in the form $\lambda_1 u_1 \oplus \cdots \oplus \lambda_k u_k$, where $u_1, \ldots, u_k$ are any elements of K, $\lambda_1, \ldots, \lambda_k$ are nonnegative real numbers satisfying $\sum_{i=1}^{k} \lambda_i = 1$ and $k = 2, 3, \ldots$.

Lemma 3.5. *Let K be a relatively compact subset of* $(\mathbf{F}(Y), d_\infty)$. *Then co*$(K)$ *is also relatively compact in* $(\mathbf{F}(Y), d_\infty)$.

Proof. (Step 1) Let A be a compact subset of Y such that $L_0u \subset A$ for all $u \in K$. Then the closed convex hull $\overline{co}(A)$ is compact subset of Y. If $v \in co(K)$, then there exist $u_1, \ldots, u_k \in K$ and nonnegative real numbers $\lambda_1, \ldots, \lambda_k$ satisfying $\sum_{i=1}^{k} \lambda_i = 1$ such that $v = \lambda_1 u_1 \oplus \cdots \oplus \lambda_k u_k$. Thus, $L_0 v = \lambda_1 L_0 u_1 \oplus \cdots \oplus \lambda_k L_0 u_k \subset \overline{co}(A)$ which implies that $co(K)$ is compact-supported.

(Step 2) If $v \in co(K)$ and $v = \lambda_1 u_1 \oplus \cdots \oplus \lambda_k u_k$, then

$$h(L_\alpha v, L_{\alpha-\delta} v) \le \sum_{i=1}^{k} \lambda_i h(L_\alpha u_i, L_{\alpha-\delta} u_i) \le \sup_{u \in K} h(L_\alpha u, L_{\alpha-\delta} u).$$

Thus, for each $\alpha \in (0,1]$

$$\begin{aligned} & \lim_{\delta \to 0} \sup_{v \in co(K)} h(L_\alpha v, L_{\alpha-\delta} v) \\ = & \lim_{\delta \to 0} \sup_{u \in K} h(L_\alpha u, L_{\alpha-\delta} u) = 0. \end{aligned}$$

Similarly, it can be prove that for each $\alpha \in [0,1)$,

$$\lim_{\delta \to 0} \sup_{v \in co(K)} h(L_{\alpha^+} v, L_{\alpha+\delta} v) = 0.$$

Therefore, $co(K)$ is relatively compact from lemma 3.3. □

Lemma 3.6. *Let K be a relatively compact subset of $(\mathbf{F}(Y), d_\infty)$. Then for each $\epsilon > 0$, there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of $[0,1]$ such that*

$$\sup_{u \in K} h(L_{\alpha_k} u, L_{\alpha_{k-1}^+} u) < \epsilon$$

for all $k = 1, 2, \ldots, r$.

Proof. Let $\epsilon > 0$ be given. By applying Lemma 3.3, for each $0 < \alpha < 1$, we can take $\delta_\alpha > 0$ so that

$$\sup_{u \in K} h(L_\alpha u, L_{\alpha-\delta_\alpha} u) < \epsilon$$

and

$$\sup_{u \in K} h(L_{\alpha^+} u, L_{\alpha+\delta_\alpha} u) < \epsilon.$$

Also, we can choose $\delta_0, \delta_1 > 0$ so that

$$\sup_{u \in K} h(L_0 u, L_{\delta_0} u) < \epsilon$$

and

$$\sup_{u\in K} h(L_1 u, L_{1-\delta_1} u) < \epsilon.$$

Let $I_0 = [0, \delta_0), I_1 = (1-\delta_1, 1]$ and $I_\alpha = (\alpha - \delta_\alpha, \alpha + \delta_\alpha)$ for each $\alpha \in (0,1)$.

Then by the compactness of [0,1], there exists $\beta_1, \dots, \beta_N \in (0,1)$ such that

$$[0,1] = I_0 \cup I_1 \cup (\cup_{i=1}^N I_{\beta_i}).$$

The collection of points $\{0, \delta_0, 1-\delta_1, 1\} \cup \{\beta_i - \delta_{\beta_i}, \beta_i, \beta_i + \delta_{\beta_i} : i = 1, \dots, N\}$ forms a partition of $[0,1]$. We denote these points in ascending order by

$$0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1.$$

Then it is obvious that for all $k = 1, 2, \dots, r$,

$$\sup_{u\in K} h(L_{\alpha_k} u, L_{\alpha_{k-1}^+} u) < \epsilon.$$

□

Now for a fixed partition $\pi : 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of [0,1], we define

$$g_\pi : \mathbf{F}(Y) \to \mathbf{F}(Y) \text{ by } g_\pi(u)(x) = \sum_{k=1}^r \alpha_{k-1} I_{A_{k-1}\setminus A_k}(x) + I_{A_r}(x),$$

where $A_k = L_{\alpha_k} u$. Then it follows that

$$L_\alpha g_\pi(u) = \begin{cases} L_{\alpha_1} u, & if \ \ 0 \le \alpha \le \alpha_1 \\ L_{\alpha_k} u, & if \ \ \alpha_{k-1} < \alpha \le \alpha_k, \ k = 2, \dots, r. \end{cases}$$

From this fact, it is trivial that

$$g_\pi(u \oplus v) = g_\pi(u) \oplus g_\pi(v) \text{ and } g_\pi(\lambda u) = \lambda g_\pi(u).$$

Lemma 3.7. *Let K be a relatively compact subset of $(\mathbf{F}(Y), d_\infty)$. Then for each $\epsilon > 0$, there exists a partition π of $[0,1]$ such that*

$$\sup_{u\in K} d_\infty(u, g_\pi(u)) < \epsilon.$$

Proof. By lemma 3.6, there exists a partition $\pi : 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ of [0,1] such that

$$\sup_{u\in K} h(L_{\alpha_k} u, L_{\alpha_{k-1}^+} u) < \epsilon$$

for all $k = 1, 2, \ldots, r$. Then

$$\sup_{u \in K} d_\infty(u, g_\pi(u)) = \sup_{u \in K} \max_{1 \le k \le r} h(L_{\alpha_k} u, L_{\alpha_{k-1}^+} u) < \epsilon.$$

$\square$

Now we are in a position to prove the theorem 3.2.

Proof of Theorem 3.2. The necessity is trivial. To prove the sufficiency, let $\epsilon > 0$ be given. By strong-compactly uniform integrability of $\{\tilde{X}_n\}$, we can choose a compact subset K of $(\mathbf{F}(Y), d_\infty)$ such that

$$\int_{\{\tilde{X}_n \notin K\}} \|\tilde{X}_n\| \, dP < \epsilon/4 \quad \text{for all} \;\; n. \tag{3.1}$$

By lemmas 3.4 and 3.5, we may assume that $\tilde{0} \in K$ and K is convex, and that K contains $\overline{co}(u)$ for all $u \in K$ without loss of generality.

By lemma 3.7, we choose a partition $\pi : 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r$ of $[0, 1]$ such that

$$\sup_{u \in K} d_\infty(u, g_\pi(u)) < \epsilon/4. \tag{3.2}$$

We note that

$$\begin{aligned}
& d_\infty(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, \frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{X}_i)) \\
\le \quad & d_\infty(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, \frac{1}{n} \oplus_{i=1}^n g_\pi(\tilde{X}_i)) \\
+ \quad & d_\infty(\frac{1}{n} \oplus_{i=1}^n g_\pi(\tilde{X}_i)), \frac{1}{n} \oplus_{i=1}^n g_\pi(\overline{co}E(\tilde{X}_i))) \\
+ \quad & d_\infty(\frac{1}{n} \oplus_{i=1}^n g_\pi(\overline{co}E(\tilde{X}_i))), \frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{X}_i)) \\
= \quad & (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}).
\end{aligned}$$

For (I), we have

$$
\begin{aligned}
\text{(I)} \quad &= \max_{1\le k\le r} \sup_{\alpha_{k-1}<\alpha\le\alpha_k} h\Big(\frac{1}{n}\oplus_{i=1}^n L_\alpha \tilde{X}_i, \frac{1}{n}\oplus_{i=1}^n L_{\alpha_k}\tilde{X}_i\Big) \\
&= \max_{1\le k\le r} h\Big(\frac{1}{n}\oplus_{i=1}^n L_{\alpha_{k-1}^+}\tilde{X}_i, \frac{1}{n}\oplus_{i=1}^n L_{\alpha_k}\tilde{X}_i\Big) \\
&\le \max_{1\le k\le r} h\Big(\frac{1}{n}\oplus_{i=1}^n L_{\alpha_{k-1}^+}\tilde{X}_i, \frac{1}{n}\oplus_{i=1}^n \overline{co}E(L_{\alpha_{k-1}^+}\tilde{X}_i)\Big) \\
&\quad + \max_{1\le k\le r} h\Big(\frac{1}{n}\oplus_{i=1}^n L_{\alpha_k}\tilde{X}_i, \frac{1}{n}\oplus_{i=1}^n \overline{co}E(L_{\alpha_k}\tilde{X}_i)\Big) \\
&\quad + \max_{1\le k\le r} h\Big(\frac{1}{n}\oplus_{i=1}^n \overline{co}E(L_{\alpha_{k-1}^+}\tilde{X}_i), \frac{1}{n}\oplus_{i=1}^n \overline{co}E(L_{\alpha_k}\tilde{X}_i)\Big).
\end{aligned}
$$

The first term and second term converge to 0 a.s. as $n\to\infty$ by our assumption.

For the third term, if we denote

$$\tilde{U}_n = I_{\{\tilde{X}_n\in K\}}\tilde{X}_n,\ \tilde{V}_n = I_{\{\tilde{X}_n\notin K\}}\tilde{X}_n,$$

then

$$
\begin{aligned}
&\max_{1\le k\le r} h\Big(\frac{1}{n}\oplus_{i=1}^n \overline{co}E(L_{\alpha_{k-1}^+}\tilde{X}_i), \frac{1}{n}\oplus_{i=1}^n \overline{co}E(L_{\alpha_k}\tilde{X}_i)\Big) \\
\le\ & \max_{1\le k\le r}\frac{1}{n}\sum_{i=1}^n Eh(L_{\alpha_{k-1}^+}\tilde{X}_i, L_{\alpha_k}\tilde{X}_i) \\
\le\ & \max_{1\le k\le r}\frac{1}{n}\sum_{i=1}^n [Eh(L_{\alpha_{k-1}^+}\tilde{U}_i, L_{\alpha_k}\tilde{U}_i) + Eh(L_{\alpha_{k-1}^+}\tilde{V}_i, L_{\alpha_k}\tilde{V}_i)] \\
<\ & \epsilon/4 + \frac{2}{n}\sum_{i=1}^n E\|\tilde{V}_i\| < \epsilon \text{ by (3.1) and (3.2).}
\end{aligned}
$$

Thus, we have

$$\text{(I)} < \epsilon \quad \text{a.s.} \quad \text{as } n\to\infty.$$

For (II), we have

$$
\begin{aligned}
\text{(II)} \quad &= \max_{1\le k\le r} h\Big(\frac{1}{n}\oplus_{i=1}^n L_{\alpha_k}\tilde{X}_i, \frac{1}{n}\oplus_{i=1}^n \overline{co}L_{\alpha_k}E(\tilde{X}_i)\Big) \\
&\to 0 \text{ a.s. as } n\to\infty \quad \text{by our assumption.}
\end{aligned}
$$

Finally for (III), we first note that $\overline{co}E(\tilde{U}_i) \in K$ by the assumptions of K and so by (3.2),

$$d_\infty(\frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{U}_i)), \frac{1}{n} \oplus_{i=1}^n g_\pi[\overline{co}E(\tilde{U}_i)]) < \epsilon/4.$$

Hence we obtain

$$\begin{aligned}
\text{(III)} \quad \leq \quad & d_\infty(\frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{U}_i)), \frac{1}{n} \oplus_{i=1}^n g_\pi[\overline{co}E(\tilde{U}_i)]) \\
+ \quad & d_\infty(\frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{V}_i)), \frac{1}{n} \oplus_{i=1}^n g_\pi[\overline{co}E(\tilde{V}_i)]) \\
\leq \quad & \frac{1}{n}\| \oplus_{i=1}^n \overline{co}E(\tilde{V}_i))\| + \frac{1}{n}\| \oplus_{i=1}^n g_\pi(\overline{co}E(\tilde{V}_i))\| + \epsilon/4 \\
\leq \quad & \frac{2}{n}\sum_{i=1}^n E\|\tilde{V}_i\| + \epsilon/4 < \epsilon.
\end{aligned}$$

Therefore we conclude

$$d_\infty(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, \frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{X}_i)) \leq 2\epsilon \ a.s. \ \text{ as } \ n \to \infty.$$

This completes the proof. □

The following corollary shows that Theorem 3.1. can be modified in the case of SCUI fuzzy random variables.

Corollary 3.8. *Let $\{\tilde{X}_n\}$ be a sequence of level-wise independent and SCUI fuzzy random variables. If*

$$\sum_{n=1}^\infty \frac{1}{n^p} E\|\tilde{X}_n\|^p < \infty \text{ for some } 1 \leq p \leq 2,$$

then

$$\lim_{n\to\infty} \frac{1}{n} d_\infty(\oplus_{i=1}^n \tilde{X}_i, \oplus_{i=1}^n \overline{co}E(\tilde{X}_i) \;=\; 0 \, a.s.$$

Corollary 3.9. *Let $\{X_n\}$ be a sequence of level-wise independent and strongly tight fuzzy random variables such that*

$$\sup_n E\|\tilde{X}_n\|^p = M < \infty \text{ for some } p > 1.$$

Then

$$\lim_{n\to\infty} d_\infty(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, \frac{1}{n} \oplus_{i=1}^n \overline{co}E(\tilde{X}_i)) \;=\; 0 \ \text{ a.s..}$$

Finally, we give SLLN for integrably bounded fuzzy random variables only by assuming that $\{\frac{1}{n} \oplus_{i=1}^{n} \overline{co}E(\tilde{X}_i)\}$ is convergent with respect to d_∞.

THEOREM 3.10. *Let $\{\tilde{X}_n\}$ be a sequence of integrably bounded fuzzy random variables such that for some $v \in \mathbf{F}(Y)$,*

$$\lim_{n\to 0} d_\infty(\frac{1}{n} \oplus_{i=1}^{n} \overline{co}(E\tilde{X}_i), v) = 0.$$

Then

$$\lim_{n\to\infty} d_\infty(\frac{1}{n} \oplus_{i=1}^{n} \tilde{X}_i, \frac{1}{n} \oplus_{i=1}^{n} \overline{co}E(\tilde{X}_i)) \to 0 \ \text{ a.s.}$$

if and only if for each $\alpha \in [0,1]$,

$$\lim_{n\to\infty} h(\frac{1}{n} \oplus_{i=1}^{n} L_\alpha\tilde{X}_i, \frac{1}{n} \oplus_{i=1}^{n} \overline{co}(EL_\alpha\tilde{X}_i)) \to 0 \ \text{ a.s.}$$

and

$$\lim_{n\to\infty} h(\frac{1}{n} \oplus_{i=1}^{n} L_{\alpha^+}\tilde{X}_i, \frac{1}{n} \oplus_{i=1}^{n} \overline{co}(EL_{\alpha^+}\tilde{X}_i)) \to 0 \ \text{ a.s..}$$

Proof. The necessity is trivial. To prove the sufficiency, it suffices to prove that

$$d_\infty(\frac{1}{n} \oplus_{i=1}^{n} \tilde{X}_i, v) \ \to \ 0 \ \text{ a.s. } \ n \to \infty.$$

Let $\tilde{S}_n = \frac{1}{n}\oplus_{i=1}^{n}\tilde{X}_i$ and let $\epsilon > 0$ be given. By Lemma 3.6 (or Lemma 4 of Guan and Li [10]), there exists a partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_r = 1$ such that

$$h(L_{\alpha_k}v, L_{\alpha_{k-1}^+}v) < \epsilon/2 \quad \text{for all } \ k = 1, \ldots, r.$$

Then by our assumption, we can find a natural number N such that

$$h(\overline{co}(EL_\alpha\tilde{S}_i), L_\alpha v) < \epsilon/2 \quad \text{for all } \ \alpha \in [0,1] \ \text{ and } \ n \geq N.$$

If $0 < \alpha \leq 1$, then $\alpha_{k-1} < \alpha \leq \alpha_k$ for some k. Since $L_{\alpha_k}\tilde{S}_n \subset L_\alpha\tilde{S}_n \subset L_{\alpha_{k-1}^+}\tilde{S}_n$ and $L_{\alpha_k}v \subset L_\alpha v \subset L_{\alpha_{k-1}^+}v$, we have that for $n \geq N$,

$$
\begin{aligned}
& h(L_\alpha \tilde{S}_n, L_\alpha v) \\
\leq\ & \max[h(L_{\alpha_k}\tilde{S}_n, L_{\alpha^+_{k-1}} v), h(L_{\alpha^+_{k-1}}\tilde{S}_n, L_{\alpha_k} v)] \\
\leq\ & \max[h(L_{\alpha_k}\tilde{S}_n, L_{\alpha_k} v),\ h(L_{\alpha^+_{k-1}}\tilde{S}_n, L_{\alpha^+_{k-1}} v)] + \epsilon/2 \\
\leq\ & \max[h(L_{\alpha_k}\tilde{S}_n, \overline{co}(EL_{\alpha_k}\tilde{S}_n), h(L_{\alpha^+_{k-1}}\tilde{S}_n, \overline{co}(EL_{\alpha^+_{k-1}}\tilde{S}_n)] + \epsilon.
\end{aligned}
$$

Thus for $n \geq N$,

$$
\begin{aligned}
& d_\infty(\tilde{S}_n, v) \\
=\ & \max_{1\leq k\leq r} \sup_{\alpha_{k-1}<\alpha\leq\alpha_k} h(L_\alpha \tilde{S}_n, L_\alpha v) \\
\leq\ & \max_{1\leq k\leq r} h(L_{\alpha_k}\tilde{S}_n, \overline{co}(EL_{\alpha_k}\tilde{S}_n)) \\
& + \max_{1\leq k\leq r} h(L_{\alpha^+_{k-1}}\tilde{S}_n, \overline{co}(EL_{\alpha^+_{k-1}}\tilde{S}_n)) + \epsilon.
\end{aligned}
$$

Therefore, by assumption we obtain

$$d_\infty(\tilde{S}_n, v) \leq \epsilon \ \text{ a.s.} \quad \text{as} \quad n \to \infty.$$

This completes the proof. □

If $\{\tilde{X}_n\}$ is a sequence of level-wise independent and CUI fuzzy random variables, then for each $\alpha \in [0,1]$, $\{L_\alpha \tilde{X}_n\}$ and $\{L_{\alpha^+}\tilde{X}_n\}$ are sequences of independent and CUI random sets. Thus if

$$\sum_{n=1}^{\infty} \frac{1}{n^p} E\|\tilde{X}_n\|^p < \infty \,\text{for some}\, 1 \leq p \leq 2,$$

by strong law of large numbers for random sets obtained Taylor and Inoue [26],

$$\lim_{n\to\infty} h(\frac{1}{n} \oplus_{i=1}^n L_\alpha \tilde{X}_i, \oplus_{i=1}^n \frac{1}{n}\overline{co}(EL_\alpha \tilde{X}_i)) \to 0 \ \text{ a.s.}$$

and

$$\lim_{n\to\infty} h(\frac{1}{n} \oplus_{i=1}^n L_{\alpha^+} \tilde{X}_i, \oplus_{i=1}^n \frac{1}{n}\overline{co}(EL_{\alpha^+} \tilde{X}_i)) \to 0 \ \text{ a.s..}$$

Therefore, the above theorem is a generalization of Theorem 3.1.

Corollary 3.11. *Let $\{\tilde{X}_n\}$ be a sequence of identically distributed fuzzy random variables with $E\|\tilde{X}_1\| < \infty$.*

Then

$$\lim_{n\to\infty} d_\infty(\frac{1}{n} \oplus_{i=1}^n \tilde{X}_i, \overline{co}(E\tilde{X}_1)) \to 0 \quad a.s.$$

if and only if for each $\alpha \in [0,1]$

$$\lim_{n\to\infty} h(\frac{1}{n} \oplus_{i=1}^n L_\alpha X_i, \overline{co}(EL_\alpha X_1)) \to 0 \quad a.s.$$

and

$$\lim_{n\to\infty} h(\frac{1}{n} \oplus_{i=1}^n L_{\alpha^+} X_i, \overline{co}(EL_{\alpha^+} X_1)) \to 0 \quad a.s..$$

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