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PERTURBATION ANAYSIS FOR THE MATRIX EQUATION $X = I - A^*X^{-1}A + B^*X^{-1}B$

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Abstract. The purpose of this paper is to study the perturbation analysis of the matrix equation $X = I - A^*X^{-1}A + B^*X^{-1}B$. Based on the matrix differentiation, we give a precise perturbation bound for the positive definite solution. A numerical example is presented to illustrate the shrpness of the perturbation bound.

1. Introduction

We consider the matrix equation

$$X = Q - A^*X^{-1}A + B^*X^{-1}B, \tag{1.1}$$

where A, B are arbitrary $n \times n$ matrices. Some special cases of Equation (1.1) are problems of practical importance, such as the matrix equation $X + M^*X^{-1}M = Q$ that arises in the control theory, ladder networks, dynamic programming, stochastic filtering, statistics, and so on [5, 8, 10]. The matrix equation $X - M^*X^{-1}M = Q$ arises in the analysis of stationary Gaussian reciprocal processes over a finite interval [1, 7].

In [2], Berzig, Duan and Samet established the existence and uniqueness of a positive definite solution of (1.1) via the Bhaskar-Lakshkanthan coupled fixed point theorem([3]).

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THEOREM 1.1 ([2]). *If there exist $a, b > 0$ satisfying following conditions*

(i) $a^{-1}A^*A + aI \leq Q \leq bI$,
(ii) $bA^*A - aB^*B \leq ab(Q - aI)$,
(iii) $bB^*B - aA^*A \leq ab(bI - Q)$,
(iv) $A^*A < \frac{a^2}{2}I$, $B^*B < \frac{a^2}{2}I$

then (1.1) *has a unique solution $X \in [aI, \infty)$ and*

$$X \in [Q + b^{-1}B^*B - a^{-1}A^*A,\ Q + a^{-1}B^*B - b^{-1}A^*A].$$

The following result is immediate consequence of Theorem 1.1.

THEOREM 1.2. *If there exist $0 < a \leq \frac{2}{3}$ such that*

$$A^*A \leq \frac{a^2}{2}I, \quad B^*B \leq \frac{a^2}{2}I,$$

then the matrix equation

$$X = I - A^*X^{-1}A + B^*X^{-1}B. \tag{1.2}$$

has a unique solution $X_U \in [aI, \infty)$ and

$$X_U \in \left[I + \frac{2}{2+a}B^*B - \frac{1}{a}A^*A,\ I + \frac{1}{a}B^*B - \frac{2}{2+a}A^*A\right]. \tag{1.3}$$

In this paper, we study the perturbation analysis of the matrix equation (1.2). Based on the matrix differentiation, we firstly give a differential bound for the unique solution of (1.2) in certain set, and then use it to derive a precise perturbation bound. A numerical example is used to show that the perturbation bound is very sharp.

Throughout this paper, we write $B > 0$ ($B \geq 0$) if the matrix B is positive definite (semidefinite). If $B - C$ is positive definite (semidefinite), then we write $B > C$ ($B \geq C$). If a positive definite matrix X satisfies $B \leq X \leq C$, we denote that $X \in [B, C]$. The symbols $\lambda_1(B)$ and $\lambda_n(B)$ denote the maximal and minimal eigenvalues of an $n \times n$ Hermitian matrix B, respectively. The symbol $\|B\|$ denotes the spectral norm of the matrix B.

2. Perturbation Analysis for the Matrix equation (1.2)

Based on the matrix differentiation, we firstly give a differential bound for the unique positive definite solution X_U of (1.2), and then use it to derive a precise perturbation bound for X_U in this section.

Definition 2.1. ([6, 9]) Let $F = (f_{ij})_{mn}$, then the matrix differentiation of F is $dF = (df_{ij})_{mn}$. For example, let

$$F = \begin{pmatrix} s+t & s^2-2t \\ 2s+t^3 & t^2 \end{pmatrix}.$$

Then

$$dF = \begin{pmatrix} ds+dt & 2sds-2dt \\ 2ds+3t^2dt & 2tdt \end{pmatrix}.$$

Lemma 2.2 ([6, 9]). *The matrix differentiation has the following properties:*

(1) $dA = 0$ *for a constant matrix* A*;*
(2) $d(\alpha X) = \alpha(dX)$*, where* α *is a complex number;*
(3) $d(X+Y) = dX + dY$*;*
(4) $d(XY) = (dX)Y + X(DY)$*;*
(5) $d(X^*) = (dX)^*$*;*
(6) $d(X^{-1}) = -X^{-1}(DX)X^{-1}$.

Theorem 2.3. *If there exist* $0 < a \le \frac{1}{2}$ *such that*

$$\|A\|^2 \le \frac{a^2}{2}, \quad \|B\|^2 \le \frac{a^2}{2}, \tag{2.4}$$

then then (1.2) *has a unique solution* $X_U \in [aI, \infty)$*, and it satisfies*

$$\|dX_U\| \le \frac{2a(\|A\| \, \|dA\| + \|B\| \, \|dB\|)}{a^2 - \|A\|^2 - \|B\|^2}. \tag{2.5}$$

Proof. Since

$$\lambda_1(A^*A) \le \|A^*A\| \le \|A\|^2,$$
$$\lambda_1(B^*B) \le \|B^*B\| \le \|B\|^2$$

then

$$\begin{aligned} A^*A &\le \lambda_1(A^*A)I \le \|A^*A\|I \le \|A\|^2 I, \\ B^*B &\le \lambda_1(B^*B)I \le \|B^*B\|I \le \|B\|^2 I. \end{aligned} \tag{2.6}$$

Combining (2.4) and (2.6) we haver

$$A^*A \le \frac{a^2}{2}I, \quad B^*B \le \frac{a^2}{2}I.$$

Then by Theorem 1.2 we have that (1.2) has a unique solution X_U in $[aI, \infty)$, which satisfies

$$X_U \in \left[I + \frac{2}{2+a}B^*B - \frac{1}{a}A^*A, \; I + \frac{1}{a}B^*B - \frac{2}{2+a}A^*A \right]. \tag{2.7}$$

Since X_U is the unique solution of (1.2) in $[aI, \infty)$,

$$X_U + A^* X_U A - B^* X_U B = I. \tag{2.8}$$

It is known that the elements of X_U are differentiable functions of the elements of A and B. Differentianting (2.8), and by Lemma 2.2, we have

$$\begin{aligned} dX_U + (dA^*)X_U^{-1}A - A^*X_U^{-1}(dX_U)X_U^{-1}A + A^*X_U^{-1}(dA) \\ -(dB^*)X_U^{-1}B + B^*X_U^{-1}(dX_U)X_U^{-1}B - B^*X_U^{-1}(dB) = 0, \end{aligned}$$

which implies that
(2.9)

$$\begin{aligned} & dX_U - A^*X_U^{-1}(dX_U)X_U^{-1}A + B^*X_U^{-1}(dX_U)X_U^{-1}B \\ = \ & -(dA^*)X_U^{-1}A - A^*X_U^{-1}(dA) + (dB^*)X_U^{-1}B + B^*X_U^{-1}(dB). \end{aligned}$$

By taking spectral norm for both sides of (2.9), we have that
(2.10)

$$\begin{aligned} & \| -(dA^*)X_U^{-1}A - A^*X_U^{-1}(dA) + (dB^*)X_U^{-1}B + B^*X_U^{-1}(dB)\| \\ & \le \|(dA^*)X_U^{-1}A\| + \|A^*X_U^{-1}(dA)\| + \|(dB^*)X_U^{-1}B\| + \|B^*X_U^{-1}(dB)\| \\ & \le \|dA^*\|\|X_U^{-1}\|\|A\| + \|A^*\|\|X_U^{-1}\|\|dA\| + \|dB^*\|\|X_U^{-1}\|\|B\| \\ & \qquad + \|B^*\|\|X_U^{-1}\|\|dB\| \\ & = 2\|X_U^{-1}\|(\|dA\|\|A\| + \|dB\|\|B\|) \\ & \le \tfrac{2}{a}(\|dA\|\|A\| + \|dB\|\|B\|) \end{aligned}$$

and

$$\begin{aligned} & \|dX_U - A^*X_U^{-1}(dX_U)X_U^{-1}A + B^*X_U^{-1}(dX_U)X_U^{-1}B\| \\ & \ge \|dX_U\| - \|A^*X_U^{-1}(dX_U)X_U^{-1}A\| - \|B^*X_U^{-1}(dX_U)X_U^{-1}B\| \\ & \ge \|dX_U\| - \|A^*\|\|X_U^{-1}\|\|dX_U\|\|X_U^{-1}\|\|A\| \\ & \qquad - \|B^*\|\|X_U^{-1}\|\|dX_U\|\|X_U^{-1}\|\|B\| \\ & = \left(1 - \|A\|^2\|X_U^{-1}\|^2 - \|B\|^2\|X_U^{-1}\|^2\right)\|dX_U\| \\ & \ge \left(1 - \tfrac{\|A\|^2}{a^2} - \tfrac{\|B\|^2}{a^2}\right)\|dX_U\|. \end{aligned} \tag{2.11}$$

Due to (2.4) we have

$$1 - \frac{\|A\|^2}{a^2} - \frac{\|B\|^2}{a^2} > 0. \tag{2.12}$$

Combination (2.10),(2.11) and noting (2.12), we have

$$\left(1 - \frac{\|A\|^2}{a^2} - \frac{\|B\|^2}{a^2}\right)\|dX_U\| \le \frac{2}{a}(\|dA\|\|A\| + \|dB\|\|B\|)$$

which implies to

$$\|dX_U\| \leq \frac{2a(\|A\|\ \|dA\| + \|B\|\ \|dB\|)}{a^2 - \|A\|^2 - \|B\|^2}.$$

□

THEOREM 2.4. *Let $\tilde{A}, \tilde{B}$ be perturbed matrices of A, B in* (1.2) *and* $\Delta A = \tilde{A} - A$, $\Delta B = \tilde{B} - B$. *If there exist $0 < a \leq \frac{1}{2}$ such that*

$$\|A\|^2 \leq \frac{a^2}{2}, \quad \|B\|^2 \leq \frac{a^2}{2}, \tag{2.13}$$

$$2\|A\|\|\Delta A\| + \|\Delta A\|^2 < \frac{a^2}{2} - \|A\|^2, \tag{2.14}$$

$$2\|B\|\|\Delta B\| + \|\Delta B\|^2 < \frac{a^2}{2} - \|B\|^2, \tag{2.15}$$

then then (1.2) *and its perturbed equation*

$$\tilde{X} = I - \tilde{A}^* \tilde{X}^{-1} \tilde{A} + \tilde{B}^* \tilde{X}^{-1} \tilde{B} \tag{2.16}$$

have a unique solutions X_U and $\tilde{X}_U$ in $[aI, \infty)$, respectively, which satisfy

$$\left\|\tilde{X}_U - X_U\right\| \leq S_{err}$$

where

$$S_{err} = \frac{2a(\|A\|\|\Delta A\| + \|\Delta A\|^2 + \|B\|\|\Delta B\| + \|\Delta B\|^2)}{a^2 - (\|A\| + \|\Delta A\|)^2 - (\|B\| + \|\Delta B\|)^2}.$$

Proof. Set $A(t) = A + t\Delta A$ and $B(t) = B + t\Delta B$, $t \in [0, 1]$ then by (2.14)

$$\begin{aligned} \|A(t)\|^2 &= \|A + t\Delta A\|^2 \leq (\|A\| + t\|\Delta A\|)^2 \\ &= \|A\|^2 + 2t\|A\|\|\Delta A\| + t^2\|\Delta A\|^2 \\ &\leq \|A\|^2 + 2\|A\|\|\Delta A\| + \|\Delta A\|^2 \\ &< \|A\|^2 + \tfrac{a^2}{2} - \|A\|^2 = \tfrac{a^2}{2}, \end{aligned} \tag{2.17}$$

similarly, by (2.15) we have

$$\|B(t)\|^2 < \frac{a^2}{2}. \tag{2.18}$$

By (2.17), (2.18) and Theorem 2.3 we derive that for arbitrary $t \in [0, 1]$, the matrix equation

$$X = I - A(t)^* X^{-1} A(t) + B(t)^* X^{-1} B(t)$$

has a unique solution $X_U(t)$ in $[aI,\infty)$, especially,

$$X_U(0)=X_U,\quad X_U(1)=(\tilde{X})_U,$$

where X_U and $\tilde{X}_U$ are the unique solutions of (1.2) and (2.16), respectively.

From Theorem 2.3 it follows that

$$\begin{aligned}
\left\|\tilde{X}_U - X_U\right\| &= \|X_U(1)-X_U(0)\| = \left\|\int_0^1 dX_U(t)\right\| \leq \int_0^1 \|dX_U(t)\| \\
&\leq \int_0^1 \frac{2a(\|A(t)\|\ \|dA(t)\| + \|B(t)\|\ \|dB(t)\|)}{a^2-\|A(t)\|^2-\|B(t)\|^2} \\
&\leq \int_0^1 \frac{2a(\|A(t)\|\ \|\Delta A\|dt + \|B(t)\|\ \|\Delta B\|dt)}{a^2-\|A(t)\|^2-\|B(t)\|^2} \\
&\leq \int_0^1 \frac{2a(\|A(t)\|\ \|\Delta A\| + \|B(t)\|\ \|\Delta B\|)}{a^2-\|A(t)\|^2-\|B(t)\|^2}dt.
\end{aligned}$$

Noting that

$$\|A(t)\| = \|A+t\Delta A\| \leq \|A\| + t\|\Delta A\|,$$

$$\|B(t)\| = \|B+t\Delta B\| \leq \|B\| + t\|\Delta B\|,$$

and combining Mean Value Theorem of Integration, we have

$$\begin{aligned}
&\left\|\tilde{X}_U - X_U\right\| \\
\leq\ & \int_0^1 \frac{2a(\|A(t)\|\ \|\Delta A\| + \|B(t)\|\ \|\Delta B\|)}{a^2-\|A(t)\|^2-\|B(t)\|^2}dt \\
\leq\ & \int_0^1 \frac{2a((\|A\|+t\|\Delta A\|)\ \|\Delta A\| + (\|B\|+t\|\Delta B\|)\ \|\Delta B\|)}{a^2-(\|A\|+t\|\Delta A\|)^2-(\|B\|+t\|\Delta B\|)^2}dt \\
\leq\ & \frac{2a((\|A\|+\xi\|\Delta A\|)\ \|\Delta A\| + (\|B\|+\xi\|\Delta B\|)\ \|\Delta B\|)}{a^2-(\|A\|+\xi\|\Delta A\|)^2-(\|B\|+\xi\|\Delta B\|)^2} \\
&\times(1-0)\ \ (0<\xi<1) \\
\leq\ & \frac{2a((\|A\|+\|\Delta A\|)\ \|\Delta A\| + (\|B\|+\|\Delta B\|)\ \|\Delta B\|)}{a^2-(\|A\|+\|\Delta A\|)^2-(\|B\|+\|\Delta B\|)^2} = S_{err}.
\end{aligned}$$

□

3. Numerical Experiments

In this section, we use a numerical example to confirm the correctness of Theorem 2.4 and the precision of the perturbation bound for the unique positive definite solution X_U of (1.2).

EXAMPLE 3.1. Consider the matrix equation

$$X = I - A^*X^{-1}A + B^*X^{-1}B,$$

and its pertubed equation

$$\tilde{X} = I - \tilde{A}^*\tilde{X}^{-1}\tilde{A} + \tilde{B}^*\tilde{X}^{-1}\tilde{B}, \tag{3.19}$$

where

$$A = \begin{pmatrix} 0.1 & 0.05 & 0 \\ 0 & 0.05 & -0.02 \\ 0.05 & -0.05 & -0.05 \end{pmatrix}, \tilde{A} = A + \begin{pmatrix} 0.5 & 0.1 & -0.1 \\ -0.1 & 0.5 & 0.5 \\ -0.2 & 0.1 & -0.1 \end{pmatrix} \times 10^{-j},$$
$$B = \begin{pmatrix} -0.05 & 0.1 & 0 \\ -0.05 & 0 & -0.05 \\ 0.05 & 0 & -0.1 \end{pmatrix}, \tilde{B} = B + \begin{pmatrix} 0.1 & 0.02 & 0.05 \\ -0.2 & 0.12 & 0.14 \\ -0.25 & 0.2 & 0.26 \end{pmatrix} \times 10^{-j},$$

$j \in \mathbb{N}$.

It is easy to verify that the conditions (2.13)-(2.15) are satisfied with $a = 0.5$, then (1.2) and its perturbed equation (3.19) have unique positive definite solutions X_U and $\tilde{X}_U$, respectively. From Berzig, Duan and Samet [2] it follows that the sequence $\{X_k\}$ and $\{Y_k\}$ generated by the iterative method

$$\begin{aligned} &X_0 = 0.5I, \qquad Y_0 = 5I, \\ &X_{k+1} = I - A^*X_k^{-1}A + B^*Y_k^{-1}B \\ &Y_{k+1} = I - A^*Y_k^{-1}A + B^*X_k^{-1}B, \quad k = 0, 1, 2, \ldots. \end{aligned}$$

both convege to X_U. Choose $\tau = 1.0 \times 10^{-15}$ as the termination scalar, that is,

$$\begin{aligned} R(X_k) &= \|X_k + A^*X_k^{-1}A - B^*X_k^{-1}B - I\| \\ R(Y_k) &= \|Y_k + A^*Y_k^{-1}A - B^*Y_k^{-1}B - I\| \end{aligned}$$

and

$$R(X) = \max\{R(X_k), R(Y_K)\} \leq \tau = 1.0 \times 10^{-15}.$$

By using the iterative method we can get the computed solution X of (1.2). Since $R(X) < 1.0 \times 10^{-15}$, then the computed solution X has a very high precision. For simplicity, we write the computed solution as the unique positive definite solution X_U. Similarly, we can also get the unique positive definite solution $\tilde{X}_U$ of the perturbed equation (3.19).

Some numerical results on the perturbation bounds for the unique positive definite solution X_U are listed in Table 1. From Table 1, we see that Theorem 2.4 gives a precise perturbation bound for the unique positive definite solution of (1.2).

Table 1. Numerical results for the different value of j

j	2	3	4	5	6
$\|\tilde{X}_U - X_U\|/\|X_U\|$	1.644×10^{-3}	1.604×10^{-4}	1.600×10^{-5}	1.599×10^{-6}	1.600×10^{-7}
$S_{err}/\|X_U\|$	7.867×10^{-3}	7.356×10^{-4}	7.305×10^{-5}	7.300×10^{-6}	7.299×10^{-7}

References

[1] P. Benner and H. Fabbender, *On the solution of the rational matrix equation* $X = Q + LX^{-1}L^*$, EURASIP J. Adv. Signal Pro. **1** (2007), 1–10 .

[2] M. Berzig, X. Duan and B. Samet, *Positive definite solution of the matrix equation* $X = Q - A^*X^{-1}A + B^*X^{-1}B$ *via Bhaskar-Lakshmikantham fixed point theorem*, Math. Sci. (Springer) **6** (2012), Art. 27.

[3] T.G. Bhaskar and V. Lakshmikantham, *Fixed point theorems in partially ordered metric spaces and applications*, Nonlinear Anal. **65** (2006), 13791393.

[4] X. Duan and Q. Wang, *Perturbation analysis for the matrix equation* $X - \sum_{i=1}^{m} A_i^* X A_i + \sum_{j=1}^{n} B_j^* X B_j = I$, J. Appl. Math. **2012** (2012), Art. ID 784620.

[5] J.C. Engwerda, A.C.M. Ran, and A.L. Rijkeboer, *Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation* $X + A^*X^{-1}A = Q$., Linear Algebra Appl. **186** (1993), 255–275.

[6] B.R. Fang, J.D. Zhou, and Y.M. Li, *Matrix Theory*, Tsinghua University Press, Beijing, China, 2006.

[7] A. Ferrante and B.C. Levy, *Hermitian solutions of the equations* $X = Q + NX^{-1}N^*$, Linear Algebra Appl. **247** (1996), 359–373.

[8] V.I. Hasanov, *Positive definite solutions of the matrix equations* $X \pm A^*X^{-q}A = Q$, Linear Algebra Appl. **404** (2005), 166–182.

[9] T.P. Minka, *Old and new matrix algebra useful for statistics*, December 2000. Notes.

[10] X. Zhan, *Computing the extreme positive definite solutions of a matrix equation*, SIAM J. Sci. Comput. **17** (1996), 632–645.

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