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t-SPLITTING SETS S OF AN INTEGRAL DOMAIN D SUCH THAT D_S IS A FACTORIAL DOMAIN

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Abstract. Let D be an integral domain, S be a saturated multiplicative subset of D such that D_S is a factorial domain, $\{X_\alpha\}$ be a nonempty set of indeterminates, and $D[\{X_\alpha\}]$ be the polynomial ring over D. We show that S is a splitting (resp., almost splitting, t-splitting) set in D if and only if every nonzero prime t-ideal of D disjoint from S is principal (resp., contains a primary element, is t-invertible). We use this result to show that $D \setminus \{0\}$ is a splitting (resp., almost splitting, t-splitting) set in $D[\{X_\alpha\}]$ if and only if D is a GCD-domain (resp., UMT-domain with $Cl(D[\{X_\alpha\}])$ torsion, UMT-domain).

1. Introduction

Let D be an integral domain with quotient field K, and let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. For each $I \in \mathbf{F}(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup\{I_v \mid J \in \mathbf{F}(D), J \subseteq I$, and J is finitely generated$\}$. An ideal $I \in \mathbf{F}(D)$ is called a *t-ideal* if $I_t = I$, and a t-ideal is a *maximal t-ideal* if it is maximal among proper integral t-ideals. It is well known that each nonzero principal

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ideal is a t-ideal; each proper integral t-ideal is contained in a maximal t-ideal; a prime ideal minimal over a t-ideal is a t-ideal; and each maximal t-ideal is a prime ideal. We say that an $I \in \mathbf{F}(D)$ is *t-invertible* if $(II^{-1})_t = D$; equivalently, if $II^{-1} \nsubseteq P$ for every maximal t-ideal P of D. Let $T(D)$ be the group of t-invertible fractional t-ideals of D under the t-multiplication $A * B = (AB)_t$, and let $Prin(D)$ be its subgroup of principal fractional ideals. The *(t-)class group* of D is an abelian group $Cl(D) = T(D)/Prin(D)$. The readers can refer to [12] for any undefined notation or terminology.

Let S be a saturated multiplicative subset of an integral domain D. As in [3], we say that S is a *t-splitting set* if for each $0 \neq d \in D$, we have $dD = (AB)_t$ for some integral ideals A, B of D, where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. We say that S is an *almost splitting set* of D if for each $0 \neq d \in D$, there is an integer $n = n(d) \geq 1$ such that $d^n = sa$ for some $s \in S$ and $a \in N(S)$, where $N(S) = \{0 \neq x \in D | (x, s')_t = D$ for all $s' \in S\}$. A *splitting set* is an almost splitting set in which $n = n(d) = 1$ for every $0 \neq d \in D$. Let $\overline{S}$ be the saturation of a multiplicative set S of D. Note that a splitting set is saturated, while both t-splitting sets and almost splitting sets need not be saturated. Also, note that S is t-splitting (resp., almost splitting) if and only if $\overline{S}$ is; so we always assume that S is saturated. It is known that an almost splitting set is t-splitting [7, Proposition 2.3]; hence

$$\text{splitting set} \Rightarrow \text{almost splitting} \Rightarrow t\text{-splitting set}.$$

Moreover, if $Cl(D)$ is torsion, then a t-splitting set is almost splitting [7, Corollary 2.4] and if $Cl(D) = 0$, then splitting set $\Leftrightarrow$ almost splitting $\Leftrightarrow$ t-splitting set.

Let X be an indeterminate over D and $D[X]$ be the polynomial ring over D. An *upper to zero* in $D[X]$ is a nonzero prime ideal Q of $D[X]$ with $Q \cap D = (0)$, and D is called a *UMT-domain* if each upper to zero in $D[X]$ is a maximal t-ideal of $D[X]$. We say that D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. As in [15], we say that D is an *almost GCD-domain* (AGCD-domain) if for each $0 \neq a, b \in D$, there is an integer $n = n(a, b) \geq 1$ such that $a^nD \cap b^nD$ is principal. Clearly, GCD-domains are AGCD-domains. It is known that AGCD-domains are UMT-domains with torsion class group [5, Lemma 3.1]; D is a PvMD if and only if D is an integrally closed UMT-domain [13, Proposition 3.2]; and D is a

GCD-domain if and only if D is a PvMD and $Cl(D) = 0$ [6, Corollary 1.5].

In [9, Theorem 2.8], the authors proved that if D_S is a principal ideal domain (PID), then S is a t-splitting set of D if and only if every nonzero prime ideal of D disjoint from S is t-invertible. They used this result to show that $D \setminus \{0\}$ is a t-splitting set of $D[X]$ if and only if D is a UMT-domain [9, Corollary 2.9]. Also, in [8, Theorem 2], the author showed that if D_S is a PID, then S is an almost splitting set of D if and only if every nonzero prime ideal of D disjoint from S contains a primary element. (A nonzero element $a \in D$ is said to be *primary* if aD is a primary ideal.) The purpose of this paper is to show that the results of [9, Theorem 2.8] and [8, Theorem 2] are also true when D_S is a factorial domain (note that a PID is a factorial domain). Precisely, we show that if D_S is a factorial domain, then S is a splitting (resp., almost splitting, t-splitting) set in D if and only if every nonzero prime t-ideal of D disjoint from S is principal (resp., contains a primary element, is t-invertible). Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D, and note that $D[\{X_\alpha\}]_{D\setminus\{0\}}$ is a factorial domain. Hence, we then use the results we obtained in this paper to show that $D \setminus \{0\}$ is a splitting (resp., almost splitting, t-splitting) set in $D[\{X_\alpha\}]$ if and only if D is a GCD-domain (resp., UMT-domain and $Cl(D[\{X_\alpha\}])$ is torsion, UMT-domain).

2. Main Results

Let D be an integral domain, $D^* = D \setminus \{0\}$, $\{X_\alpha\}$ be a nonempty set of indeterminates over D, and $D[\{X_\alpha\}]$ be the polynomial ring over D.

We begin this section with nice characterizations of splitting sets, almost splitting sets, and t-splitting sets which appear in [2, Theorem 2.2], [4, Proposition 2.7], and [3, Corollary 2.3], respectively.

Lemma 1. *Let S be a saturated multiplicative subset of D.*

1. *S is a splitting (resp., t-splitting) set of D if and only if $dD_S \cap D$ is principal (resp., t-invertible) for every $0 \neq d \in D$.*
2. *S is an almost splitting set of D if and only if for every $0 \neq d \in D$, there is a positive integer $n = n(d)$ such that $d^n D_S \cap D$ is principal.*

Note that if D_S is a PID, then every nonzero prime ideal P of D disjoint from S has height-one, and thus P is a t-ideal. Hence, our first result is a generalization of [9, Theorem 2.8] that if D_S is a PID, then S is a t-splitting set in D if and only if every nonzero prime ideal of D disjoint from S is t-invertible. The proof is similar to those of [9, Theorem 2.8] and [8, Theorem 2].

THEOREM 2. *Let D be an integral domain and S be a saturated multiplicative subset of D such that D_S is a factorial domain. Then S is a t-splitting set in D if and only if every prime t-ideal of D disjoint from S is t-invertible.*

Proof. ($\Rightarrow$) Assume that S is a t-splitting set of D, and let P be a prime t-ideal of D with $P \cap S = \emptyset$. Then $(PD_S)_t = PD_S$ [3, Theorem 4.9], and hence $PD_S = pD_S$ for some $p \in P$ since D_S is a factorial domain. Thus, by Lemma 1, $P = PD_S \cap D = pD_S \cap D$ is t-invertible.

($\Leftarrow$) Let $0 \neq d \in D$. Then $dD_S = p_1^{e_1} \cdots p_k^{e_k} D_S$ for some $p_i \in D$ and positive integers e_i such that every p_i is a prime element in D_S and $p_i D_S \neq p_j D_S$ if $i \neq j$. Let P_i be the prime ideal of D such that $P_i D_S = p_i D_S$. Clearly, P_i is minimal over dD, and hence P_i is a t-ideal. Moreover, $P_i \cap S = \emptyset$; so P_i is t-invertible by assumption (and hence a maximal t-ideal [13, Proposition 1.3]). Note that $(P_i^{e_i})_t$ is P_i-primary [1, Lemma 1] because P_i is a maximal t-ideal. Also, $(P_i^{e_i})_t D_S = p_i^{e_i} D_S$, and thus $P_i^{e_i} D_S \cap D = (P_i^{e_i})_t$ and $(P_i^{e_i})_t$ is t-invertible. Hence

$$\begin{aligned} dD_S \cap D &= p_1^{e_1} \cdots p_k^{e_k} D_S \cap D \\ &= (p_1^{e_1} D_S \cap \cdots \cap p_k^{e_k} D_S) \cap D \\ &= (P_1^{e_1} D_S \cap \cdots \cap P_k^{e_k} D_S) \cap D \\ &= (P_1^{e_1} D_S \cap D) \cap \cdots \cap (P_k^{e_k} D_S \cap D) \\ &= (P_1^{e_1})_t \cap \cdots \cap (P_k^{e_k})_t \\ &= ((P_1^{e_1})_t \cdots (P_k^{e_k})_t)_t. \end{aligned}$$

Thus, S is a t-splitting set by Lemma 1. □

The next result is a generalization of [9, Corollary 2.9] that D^* is a t-splitting set in $D[X]$, where X is an indeterminate over D, if and only if D is a UMT-domain.

COROLLARY 3. *D^* is a t-splitting set in $D[\{X_\alpha\}]$ if and only if D is a UMT-domain.*

Proof. ($\Rightarrow$) Let $X \in \{X_\alpha\}$, and let P be a nonzero prime ideal of $D[X]$ with $P \cap D = (0)$. Then P is a prime t-ideal of $D[X]$, and hence

$Q := P[Y]$, where $Y = \{X_\alpha\} \setminus \{X\}$, is a prime t-ideal of $D[\{X_\alpha\}]$ [11, Lemma 2.1(1)] such that $Q \cap D^* = \emptyset$. Hence, Q is t-invertible by Theorem 2 because $D[\{X_\alpha\}]_{D^*}$ is a factorial domain. Note that $D[\{X_\alpha\}] = (QQ^{-1})_t = ((P[Y])(P[Y])^{-1})_t = ((P[Y])(P^{-1}[Y])_t = ((PP^{-1})[Y])_t = (PP^{-1})_t[Y]$ [11, Lemma 2.1(1)]. Hence, P is t-invertible, and thus P is a maximal t-ideal of $D[X]$.

($\Leftarrow$) Let Q be a prime t-ideal of $D[\{X_\alpha\}]$ such that $Q \cap D^* = \emptyset$. Since $Q \neq (0)$, there are $X_1, \ldots, X_n \in \{X_\alpha\}$ such that $Q \cap D[X_1, \ldots, X_{n-1}] = (0)$, but $Q \cap D[X_1, \ldots, X_n] \neq (0)$. Let $R = D[X_1, \ldots, X_{n-1}]$ and $P = Q \cap R[X_n]$. Then R is a UMT-domain [11, Theorem 2.4] and P is an upper to zero in $R[X_n]$. Hence, P is a t-invertible prime t-ideal. Let $Z = \{X_\alpha\} \setminus \{X_1, \ldots, X_n\}$, and note that $P[Z] \subseteq Q$ and $P[Z]$ is a t-invertible prime t-ideal of $D[\{X_\alpha\}]$ (see the proof of ($\Rightarrow$) above). Hence, $P[Z]$ is a maximal t-ideal of $D[\{X_\alpha\}]$, and thus $Q = P[Z]$ and Q is t-invertible. Thus, by Theorem 2, D^* is a t-splitting set. $\square$

We next give an almost splitting set analog of Theorem 2. Even though the proof is a word for word translation of the proof of [8, Theorem 2], we give it for the completeness of this paper.

Theorem 4. *Let D be an integral domain and S be a saturated multiplicative subset of D such that D_S is a factorial domain. Then S is an almost splitting set in D if and only if every prime t-ideal of D disjoint from S contains a primary element.*

Proof. ($\Rightarrow$) Assume that S is an almost splitting set of D, and let P be a prime t-ideal of D disjoint from S. Then $PD_S = pD_S$ for some $p \in P$ (see the proof of Theorem 2), and since S is almost splitting, by Lemma 1, there is a positive integer n such that $P \supseteq p^n D_S \cap D = qD$ for some $q \in D$. Clearly, q is a primary element. Thus, P contains a primary element q.

($\Leftarrow$) Let $0 \neq d \in D$. Then $dD_S = p_1^{e_1} \cdots p_k^{e_k} D_S$, where every e_i is a positive integer and the p_i's are non-associate prime elements in D_S (see the proof of Theorem 2). Let P_i be the prime ideal of D such that $P_i D_S = p_i D_S$. Then P_i is a prime t-ideal of D and $P_i \cap S = \emptyset$; so P_i contains a primary element q_i. Clearly, $q_i D_S = p_i^{n_i} D_S$ for some positive integer n_i. Let $n = n_1 \cdots n_k$ and $m_i = \frac{n}{n_i} e_i$. Then $p_i^{ne_i} D_S = q_i^{m_i} D_S$, and hence

$$\begin{aligned} d^n D_S \cap D &= ((p_1^{ne_1})D_S \cap \cdots \cap (p_k^{ne_k})D_S) \cap D \\ &= ((q_1^{m_1} D_S) \cap \cdots \cap (q_k^{m_k} D_S)) \cap D \\ &= (q_1^{m_1} D_S \cap D) \cap \cdots \cap (q_k^{m_k} D_S \cap D) \\ &= (q_1^{m_1})D \cap \cdots \cap (q_k^{m_k})D \\ &= (q_1^{m_1} \cdots q_k^{m_k})D, \end{aligned}$$

where the fourth and last equalities follow from the fact that each $q_i^{m_i}$ is a primary element with $\sqrt{q_i^{m_i} D} \neq \sqrt{q_j^{m_j} D}$ for $i \neq j$. Therefore, S is an almost splitting set by Lemma 1. □

Let $N(D^*) = \{f \in D[\{X_\alpha\}] \mid (f,d)_v = D[\{X_\alpha\}]$ for all $d \in D^*\}$. It is clear that $(f,d)_v = D[\{X_\alpha\}]$ for all $d \in D^*$ if and only if $c(f)_v = D$, where $c(f)$ is the ideal of D generated by the coefficients of f. Hence, $Cl(D[\{X_\alpha\}]_{N(D^*)}) = 0$ [14, Theorem 2.14]. The next result is a generalization of [5, Theorem 2.4].

Corollary 5. *D^* is an almost splitting set in $D[\{X_\alpha\}]$ if and only if D is a UMT-domain and $Cl(D[\{X_\alpha\}])$ is torsion.*

Proof. ($\Rightarrow$) If D^* is an almost splitting set in $D[\{X_\alpha\}]$, then $Cl(D[\{X_\alpha\}]_{D^*}) = Cl((D[\{X_\alpha\}])_{N(D^*)}) = 0$. Thus, $Cl(D[\{X_\alpha\}])$ is torsion [7, Theorem 2.10(2)]. Also, since almost splitting sets are t-splitting sets, D is a UMT-domain by Corollary 3.

($\Leftarrow$) Assume that D is a UMT-domain and $Cl(D[\{X_\alpha\}])$ is torsion. Then D^* is a t-splitting set by Corollary 3, and since $Cl(D[\{X_\alpha\}])$ is torsion, D^* is an almost splitting set. □

Corollary 6. *If D is integrally closed, then D^* is an almost splitting (resp., a t-splitting) set in $D[\{X_\alpha\}]$ if and only if D is an AGCD-domain (resp., a PvMD).*

Proof. Note that $Cl(D[\{X_\alpha\}]) = Cl(D)$ [10, Corollary 2.13]; an integrally closed UMT-domain is a PvMD; and an integrally closed AGCD-domain is a PvMD with torsion class group. Hence, the result follows directly from Corollaries 3 and 5. □

Theorem 7. *Let D be an integral domain and S be a saturated multiplicative subset of D such that D_S is a factorial domain. Then S is a splitting set in D if and only if every prime t-ideal of D disjoint from S is principal.*

Proof. ($\Rightarrow$) Let P be a prime t-ideal of D with $P \cap S = \emptyset$. Then $PD_S = pD_S$ for some prime element p of D_S (see the proof of Theorem 2), and thus $PD_S \cap D = pD_S \cap D$ is principal by Lemma 1.

($\Leftarrow$) An argument similar to the proof ($\Leftarrow$) of Theorem 4 shows that $dD_S \cap D$ is principal for every $0 \neq d \in D$. Thus, by Lemma 1, S is a splitting set. $\square$

Let X be an indeterminate over D. In [9, p. 77] (cf. [2, Example 4.7]), it was noted that D^* is a splitting set in $D[X]$ if and only if D is a GCD-domain.

COROLLARY 8. *D^* is a splitting set in $D[\{X_\alpha\}]$ if and only if D is a GCD-domain.*

Proof. If D^* is a splitting set in $D[\{X_\alpha\}]$, then $Cl(D) = Cl(D[\{X_\alpha\}]) = 0$ [2, Corollary 3.8] because $Cl(D[\{X_\alpha\}]_{D^*}) = Cl((D[\{X_\alpha\}])_{N(D^*)}) = 0$. Hence, D is integrally closed [10, Corollary 2.13] and D is a UMT-domain by Corollary 3. Thus, D is a GCD domain because D is an integrally closed UMT-domain with $Cl(D) = 0$. Conversely, assume that D is a GCD-domain. Then D^* is a t-splitting set in $D[\{X_\alpha\}]$ by Corollary 3 and $Cl(D[\{X_\alpha\}]) = Cl(D) = 0$. Thus, D^* is a splitting set. $\square$

Let S be a saturated multiplicative subset of an integral domain D such that D_S is a factorial domain. The proofs of Theorems 2, 4, and 7 show that S is splitting (resp., almost splitting, t-splitting) if and only if for every nonzero prime element p of D_S, the ideal $pD_S \cap D$ is principal (resp., contains a primary element, t-invertible).

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