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JOIN-MEET APPROXIMATION OPERATORS INDUCED BY ALEXANDROV FUZZY TOPOLOGIES

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Abstract. In this paper, we investigate the properties of Alexandrov fuzzy topologies and join-meet approximation operators. We study fuzzy preorder, Alexandrov topologies join-meet approximation operators induced by Alexandrov fuzzy topologies. We give their examples.

1. Introduction

Pawlak [8,9] introduced rough set theory as a formal tool to deal with imprecision and uncertainty in data analysis. Hájek [2] introduced a complete residuated lattice which is an algebraic structure for many valued logic. Radzikowska [10] developed fuzzy rough sets in complete residuated lattice. Bělohlávek [1] investigated information systems and decision rules in complete residuated lattices. Zhang [6,7] introduced Alexandrov L-topologies induced by fuzzy rough sets. Kim [5] investigated the properties of Alexandrov topologies in complete residuated lattices. Höhle [3] introduced L-fuzzy topologies and L-fuzzy interior approximation operators on complete residuated lattices.

In this paper, we investigate the properties of Alexandrov fuzzy topologies and join-meet approximation operators in a sense as Höhle [3]. We study fuzzy preorder, Alexandrov topologies join-meet approximation

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operators induced by Alexandrov fuzzy topologies. We give their examples.

2. Preliminaries

Definition 2.1. [1-3] A structure $(L, \vee, \wedge, \odot, \to, \bot, \top)$ is called a *complete residuated lattice* iff it satisfies the following properties:
(L1) $(L, \vee, \wedge, \bot, \top)$ is a complete lattice where $\bot$ is the bottom element and $\top$ is the top element;
(L2) $(L, \odot, \top)$ is a monoid;
(L3) It has an adjointness,i.e.

$$x \leq y \to z \text{ iff } x \odot y \leq z.$$

An operator $^* : L \to L$ defined by $a^* = a \to \bot$ is called *strong negations* if $a^{**} = a$.

$$\top_x(y) = \begin{cases} \top, & \text{if } y = x, \\ \bot, & \text{otherwise.} \end{cases} \quad \top^*_x(y) = \begin{cases} \bot, & \text{if } y = x, \\ \top, & \text{otherwise.} \end{cases}$$

In this paper, we assume that $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *.

Definition 2.2. [6,7] Let X be a set. A function $e_X : X \times X \to L$ is called a fuzzy preorder if it satisfies the following conditions
(E1) reflexive if $e_X(x, x) = 1$ for all $x \in X$,
(E2) transitive if $e_X(x, y) \odot e_X(y, z) \leq e_X(x, z)$, for all $x, y, z \in X$'

Example 2.3. (1) We define a function $e_L : L \times L \to L$ as $e_L(x, y) = x \to y$. Then e_L is a fuzzy preorder on L.
(2) We define a function $e_{L^X} : L^X \times L^X \to L$ as $e_{L^X}(A, B) = \bigwedge_{x \in X}(A(x) \to B(x))$. Then e_{L^X} is a fuzzy preorder from Lemma 2.4 (9).

Lemma 2.4. [1,2] *Let $(L, \vee, \wedge, \odot, \to, ^*, \bot, \top)$ be a complete residuated lattice with a strong negation *. For each $x, y, z, x_i, y_i \in L$, the following properties hold.*
(1) *If $y \leq z$, then $x \odot y \leq x \odot z$.*
(2) *If $y \leq z$, then $x \to y \leq x \to z$ and $z \to x \leq y \to x$.*
(3) *$x \to y = \top$ iff $x \leq y$.*
(4) *$x \to \top = \top$ and $\top \to x = x$.*
(5) *$x \odot y \leq x \wedge y$.*

(6) $x \odot (\bigvee_{i\in\Gamma} y_i) = \bigvee_{i\in\Gamma}(x \odot y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \odot y = \bigvee_{i\in\Gamma}(x_i \odot y)$.

(7) $x \to (\bigwedge_{i\in\Gamma} y_i) = \bigwedge_{i\in\Gamma}(x \to y_i)$ and $(\bigvee_{i\in\Gamma} x_i) \to y = \bigwedge_{i\in\Gamma}(x_i \to y)$.

(8) $\bigvee_{i\in\Gamma} x_i \to \bigvee_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$ and $\bigwedge_{i\in\Gamma} x_i \to \bigwedge_{i\in\Gamma} y_i \geq \bigwedge_{i\in\Gamma}(x_i \to y_i)$.

(9) $(x \to y) \odot x \leq y$ and $(y \to z) \odot (x \to y) \leq (x \to z)$.

(10) $x \to y \leq (y \to z) \to (x \to z)$ and $x \to y \leq (z \to x) \to (z \to y)$.

(11) $\bigwedge_{i\in\Gamma} x_i^* = (\bigvee_{i\in\Gamma} x_i)^*$ and $\bigvee_{i\in\Gamma} x_i^* = (\bigwedge_{i\in\Gamma} x_i)^*$.

(12) $(x \odot y) \to z = x \to (y \to z) = y \to (x \to z)$ and $(x \odot y)^* = x \to y^*$.

(13) $x^* \to y^* = y \to x$ and $(x \to y)^* = x \odot y^*$.

(14) $y \to z \leq x \odot y \to x \odot z$.

Definition 2.5. [5] A map $\mathcal{K} : L^X \to L^Y$ is called a *join-meet operator* if it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,

(K1) $\mathcal{K}(\alpha \odot A) = \alpha \to \mathcal{K}(A)$ where $(\alpha \odot A)(x) = \alpha \odot A(x)$ for each $x \in X$,

(K2) $\mathcal{K}(\bigvee_{i\in I} A_i) = \bigwedge_{i\in I} \mathcal{K}(A_i)$,

(K3) $\mathcal{K}(A) \leq A^*$,

(K4) $\mathcal{K}(\mathcal{K}^*(A)) \geq \mathcal{K}(A)$.

Definition 2.6. [4] An operator $\mathbf{T} : L^X \to L$ is called an *Alexandrov fuzzy topology* on X iff it satisfies the following conditions, for all $A, A_i \in L^X$, and $\alpha \in L$,

(T1) $\mathbf{T}(\alpha_X) = \top$ where $\alpha_X(x) = \alpha$,

(T2) $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$ and $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$,

(T3) $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$, where $(\alpha \odot A)(x) = \alpha \odot A(x)$ for each $x \in X$,

(T4) $\mathbf{T}(\alpha \to A) \geq \mathbf{T}(A)$.

Definition 2.7. [5] A subset $\tau \subset L^X$ is called an *Alexandrov topology* if it satisfies satisfies the following conditions.

(O1) $\alpha_X \in \tau$.

(O2) If $A_i \in \tau$ for $i \in \Gamma$, $\bigvee_{i\in\Gamma} A_i, \bigwedge_{i\in\Gamma} A_i \in \tau$.

(O3) $\alpha \odot A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

(O4) $\alpha \to A \in \tau$ for all $\alpha \in L$ and $A \in \tau$.

Remark 2.8. (1) *If $\mathbf{T} : L^X \to L$ is an Alexandrov fuzzy topology. Define $\mathbf{T}^*(A) = \mathbf{T}(A^*)$. Then $\mathbf{T}^*$ is an Alexandrov fuzzy topology.*

(2) *If $\mathbf{T}$ is an Alexandrov fuzzy topology on X, $\tau_T^r = \{A \in L^X \mid \mathbf{T}(A) \geq r\}$ is an Alexandrov topology on X and $\tau_T^r \subset \tau_T^s$ for $s \leq r \in L$.*

(3) *If* $\mathbf{T}^*$ *is an Alexandrov fuzzy topology on* X, $(\tau_T^r)^* = \{A \in L^X \mid \mathbf{T}^*(A) \geq r\}$ *is an Alexandrov topology on* X *and* $(\tau_T^r)^* = \tau_{T^*}^r$.

3. Join-meet approximation operators induced by Alexandrov fuzzy topologies

THEOREM 3.1. *If* $\mathcal{K}$ *is a join-meet approximation operator, then* $\tau_{\mathcal{K}} = \{A \in L^X \mid \mathcal{K}(A) = A^*\}$ *is an Alexandrov topology on* X.

Proof. (O1) Since $\mathcal{K}(\top_X) = \perp_X$ and $\mathcal{K}(\alpha \odot \top_X) = \alpha \to \mathcal{K}(\top_X) = \alpha_X^*$, then $\alpha_X^* = \mathcal{K}(\alpha_X)$. Thus $\alpha_X \in \tau_{\mathcal{K}}$.

(O2) For $A_i \in \tau_{\mathcal{K}}$ for each $i \in \Gamma$, by (K2), $\mathcal{K}(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \mathcal{K}(A_i) = \bigwedge_{i\in\Gamma} A_i^*$. Then $\bigvee_{i\in\Gamma} A_i \in \tau_{\mathcal{K}}$. Since $\mathcal{K}$ is decreasing function, $\bigvee_{i\in\Gamma} A_i^* = \bigvee_{i\in\Gamma} \mathcal{K}(A_i) = \mathcal{K}(\bigwedge_{i\in\Gamma} A_i) \leq (\bigwedge_{i\in\Gamma} A_i)^*$, Thus, $\bigvee_{i\in\Gamma} A_i \in \tau_{\mathcal{K}}$.

(O3) For $A \in \tau_{\mathcal{K}}$, $\mathcal{K}(\alpha \odot A) = \alpha \to \mathcal{K}(A) = (\alpha \odot A)^*$. Then $\alpha \odot A \in \tau_{\mathcal{K}}$.

(O4) For $A \in \tau_{\mathcal{K}}$, since $\alpha \odot (\alpha \to A) \leq A$, then $\alpha \to \mathcal{K}(\alpha \to A) = \mathcal{K}(\alpha \odot (\alpha \to A)) \geq \mathcal{K}(A)$. So, $\alpha \odot \mathcal{K}(A) \leq \mathcal{K}(\alpha \to A) \leq (\alpha \to A)^* = \alpha \odot A^*$. Thus $(\alpha \to A) \in \tau_{\mathcal{K}}$. □

THEOREM 3.2. *Let* $\mathbf{T}$ *be an Alexandrov fuzzy topology on* X. *Define*

$$R_T^r(x,y) = \bigwedge\{A(x) \to A(y) \mid \mathbf{T}(A) \geq r\}.$$

Then the following properties hold.

(1) R_T^r *is a fuzzy preorder with* $R_T^r \leq R_T^s$ *for each* $r \leq s$.

(2) *Define* $\mathcal{K}_{R_T^{r*}} : L^X \to L^X$ *as follows*

$$\mathcal{K}_{R_T^{r*}}(A)(y) = \bigwedge_{x\in X} (A(x) \to R_T^{r*}(x,y)).$$

Then $\mathcal{K}_{R_T^{r*}}$ *is a join-meet operator on* X *with* $\mathcal{K}_{R_T^{s*}} \leq \mathcal{K}_{R_T^{r*}}$ *for each* $r \leq s$.

(3) $\tau_T^r = \tau_{\mathcal{K}_{R_T^{r*}}}$.

Proof. (1) Since $\mathbf{T}(B) \geq r$ iff $B \in \tau_T^r$, then $R_T^r(x,y) = \bigwedge_{B\in\tau_T^r}(B(x) \to B(y))$. Since $R_T^r(x,x) = \bigwedge_{B\in\tau_T^r}(B(x) \to B(x)) = \top$ and

$$\begin{aligned} &R_T^r(x,y) \odot R_T^r(y,z) = \textstyle\bigwedge_{B\in\tau_T^r}(B(x) \to B(y)) \odot \bigwedge_{B\in\tau_T^r}(B(y) \to B(z)) \\ &\leq \textstyle\bigwedge_{B\in\tau_T^r}(B(x) \to B(y)) \odot (B(y) \to B(z)) \\ &\leq \textstyle\bigwedge_{B\in\tau_T^r}(B(x) \to B(z)) = R_T^r(x,y). \end{aligned}$$

Hence R^r_T is a fuzzy preorder. For $r \leq s$, since $\mathbf{T}(B) \geq s \geq r$, we have $R^r_T \leq R^s_T$.

(2) (K1)

$$\begin{aligned} \mathcal{K}_{R^{r*}_T}(\alpha \odot A)(y) &= \bigwedge_{x\in X}((\alpha \odot A)(x) \to R^{r*}_T(x,y)) \\ &= \alpha \to \bigwedge_{x\in X}(A(x) \to R^{r*}_T(x,y)) = \alpha \to \mathcal{K}_{R^{r*}_T}(A)(y). \end{aligned}$$

(K2)

$$\begin{aligned} &\mathcal{K}_{R^{r*}_T}(\bigvee_{i\in\Gamma} A_i)(y) = \bigwedge_{x\in X}(\bigvee_{i\in\Gamma} A_i(x) \to R^{r*}_T(x,y)) \\ &= \bigwedge_{i\in\Gamma}\bigwedge_{x\in X}(A_i(x) \to R^{r*}_T(x,y)) = \bigwedge_{i\in\Gamma}\mathcal{K}_{R^{r*}_T}(A_i)(y). \end{aligned}$$

(K3) $\mathcal{K}_{R^{r*}_T}(A)(y) = \bigwedge_{x\in X}(A(x) \to R^{r*}_T(x,y)) \leq A(x) \to R^{r*}_T(x,x) = A(x) \to \perp = A^*(x)$.

(K4)

$$\begin{aligned} \mathcal{K}_{R^{r*}_T}(\mathcal{K}^*_{R^{r*}_T}(A))(z) &= \bigwedge_{y\in X}(\mathcal{K}^*_{R^{r*}_T}(A)(y) \to R^{r*}_T(y,z)) \\ &= \bigwedge_{y\in X}(\bigwedge_{x\in X}(A(x) \to R^{r*}_T(x,y))^* \to R^{r*}_T(y,z)) \\ &= \bigwedge_{y\in X}(\bigvee_{x\in X}(A(x) \odot R^r_T(x,y)) \to R^{r*}_T(y,z)) \\ &= \bigwedge_{x,y\in X}(A(x) \to (R^r_T(x,y)) \to R^{r*}_T(y,z))) \\ &= \bigwedge_{x\in X}(A(x) \to \bigwedge_{y\in X}(R^r_T(x,y)) \to R^{r*}_T(y,z))) \\ &= \bigwedge_{x\in X}(A(x) \to (\bigvee_{y\in X}(R^r_T(x,y)) \odot R^r_T(y,z))^*) \\ &\geq \bigwedge_{x\in X}(A(x) \to R^{r*}_T(x,z)) \\ &= \mathcal{K}_{R^{r*}_T}(A)(z). \end{aligned}$$

Hence $\mathcal{K}_{R^{r*}_T}$ is a join-meet operator on X. For $r \leq s$, since $R^r_T \leq R^s_T$, then $\mathcal{K}_{R^{s*}_T} \leq \mathcal{K}_{R^{t*}_T}$.

(3) Let $A \in \tau^r_T$. Since $R^r_T(x,y) = \bigwedge_{B\in\tau^r_T}(B(x) \to B(y))$,

$$\begin{aligned} A^*(y) \odot R^r_T(x,y) &= A^*(y) \odot \bigwedge_{B\in\tau^r_T}(B(x) \to B(y)) \\ &\leq A^*(y) \odot (A^*(y) \to A^*(x)) \leq A^*(x). \end{aligned}$$

Thus $A^*(y) \leq R^r_T(x,y) \to A^*(x) = A(x) \to R^{r*}_T(x,y)$. Then $A^* \leq \mathcal{K}_{R^{r*}_T}(A)$. By (K3), $\mathcal{K}_{R^{r*}_T}(A) = A^*$;i.e.$A \in \tau_{\mathcal{K}_{R^{r*}_T}}$. So, $\tau^r_T \subset \tau_{\mathcal{K}_{R^{r*}_T}}$.

Let $A \in \tau_{\mathcal{K}_{R^{r*}_T}}$;i.e. $\mathcal{K}_{R^{r*}_T}(A) = A^*$. Then

$$\begin{aligned} A^* &= \bigwedge_{x\in X}(A(x) \to R^{r*}_T(x,-)) \\ &= \bigwedge_{x\in X}(A(x) \to (\bigwedge_{B\in\tau^r_T}(B(x) \to B))^*) \\ &= \bigwedge_{x\in X}(A(x) \to \bigvee_{B\in\tau^r_T}(B(x) \odot B^*)) \end{aligned}$$

Since $\bigvee_{B\in\tau^r_T}(B(x) \odot B^*) \in (\tau^r_T)^*$ and $A(x) \to \bigvee_{B\in\tau^r_T}(B(x) \odot B^*) \in (\tau^r_T)^*$, we have $A^* \in (\tau^r_T)^*$;i.e $A \in \tau^r_T$. So, $\tau_{\mathcal{K}_{R^{r*}_T}} \subset \tau^r_T$. □

THEOREM 3.3. *Let* $\mathbf{T}$ *be an Alexandrov fuzzy topology on* X. *Define*

$$R_T^{-r}(x,y) = \bigwedge\{B(y) \to B(x) \mid \mathbf{T}(B) \geq r\}.$$

Then the following properties hold.

(1) R_T^{-r} *is a fuzzy preorder with* $R_T^{-r} \leq R_T^{-s}$ *for each* $r \leq s$ *and*

$$R_T^{-r}(x,y) = R_{T^*}^{r}(x,y).$$

(2) $\mathcal{K}_{R_T^{-r*}}$ *is a join-meet operator on* X *such that*

$$\mathcal{K}_{R_T^{-r*}}(A)(y) = \bigwedge_{x\in X}(A(x) \to R_T^{-r*}(x,y)) = \bigwedge_{x\in X}(A(x) \to R_{T^*}^{r*}(x,y)).$$

(3) $(\tau_T^r)^* = \tau_{\mathcal{K}_{R_T^{-r*}}} = \tau_{\mathcal{K}_{R_{T^*}^{r*}}}$.

(4) *If* $\mathcal{K}_{R_T^{r_i*}}(A) = B$ *for all* $i \in \Gamma \neq \emptyset$, *then* $\mathcal{K}_{R_T^{s*}}(A) = B$ *with* $s = \bigvee_{i\in\Gamma} r_i$.

(5) *If* $\mathcal{K}_{R_T^{-r_i}}(A) = B$ *for all* $i \in \Gamma \neq \emptyset$, *then* $\mathcal{K}_{R_T^{-s}}(A) = B$ *with* $s = \bigvee_{i\in\Gamma} r_i$.

(6) $\mathcal{K}_{R_{T^*}^{r*}}(A) = \bigvee\{A_i \mid A_i \leq A^*,\ \mathbf{T}(A_i) \geq r\}$ *for all* $A \in L^X$ *and* $r \in L$. *Moreover,* $R_T^{-r}(x,y) = \mathcal{K}^*_{R_{T^*}^{r*}}(\top_x)(y)$, *for each* $x,y \in X$.

(7) $\mathcal{K}_{R_T^{r*}}(A) = \bigvee\{A_i \mid A_i \leq A^*,\ \mathbf{T}^*(A_i) \geq r\}$ *for all* $A \in L^X$ *and* $r \in L$. *Moreover,* $R_T^{r}(x,y) = \mathcal{K}^*_{R_T^{r*}}(\top_x)(y)$, *for each* $x,y \in X$.

Proof. (1) By a similar method as (1), R_T^{-r} is a fuzzy preorder. Moreover,

$$\begin{aligned} R_T^{-r}(x,y) &= \bigwedge\{B(y) \to B(x) \mid \mathbf{T}(B) \geq r\} \\ &= \bigwedge\{B^*(x) \to B^*(y) \mid \mathbf{T}(B^*) = \mathbf{T}^*(B) \geq r\} \\ &= R_{T^*}^{r}(x,y). \end{aligned}$$

(2) By (1), $R_T^{-r}(x,y) = \bigwedge_{B\in\tau_T^r}(B(y) \to B(x))$ is a fuzzy preorder.

(3) Let $A \in (\tau_T^r)^*$. Then $A^* \in \tau_T^r$ and

$$\begin{aligned} A^*(y) \odot R_T^{-r}(x,y) &= A^*(y) \odot \bigwedge_{B\in\tau_T^r}(B(y) \to B(x)) \\ &\leq A^*(y) \odot (A^*(y) \to A^*(x)) \leq A^*(x). \end{aligned}$$

Thus $A^*(y) \leq R_T^{-r}(x,y) \to A^*(x) = A(x) \to R_T^{-r*}(x,y)$. Hence $\mathcal{K}_{R_T^{-r*}}(A) = A^*$;i.e.$A \in \tau_{\mathcal{K}_{R_T^{-r*}}}$. So, $(\tau_T^r)^* \subset \tau_{\mathcal{K}_{R_T^{-r*}}}$.

Let $A \in \tau_{\mathcal{K}_{R_T^{-r*}}}$;i.e. $\mathcal{K}_{R_T^{-r*}}(A) = A^*$. Then

$$\begin{array}{ll} A^* & = \bigwedge_{x\in X}(A(x) \to R_T^{-r*}(x,-)) \\ & = \bigwedge_{x\in X}(A(x) \to (\bigwedge_{B\in\tau^r_{T^*}}(B(x) \to B))^*) \\ & = \bigwedge_{x\in X}(A(x) \to \bigvee_{B\in\tau^r_{T^*}}(B(x) \odot B^*)) \end{array}$$

Since $\bigvee_{B\in\tau_T^r}(B(x) \odot B^*) \in \tau_T^r$ and $A(x) \to \bigvee_{B\in\tau_T^r}(B(x) \odot B^*) \in \tau_T^r$, we have $A^* \in \tau_T^r$;i.e $A \in (\tau_T^r)^*$. So, $\tau_{\mathcal{K}_{R_T^{-r*}}} \subset (\tau_T^r)^*$.

(4) Let $\mathcal{K}_{R_T^{r_i*}}(A) = B$ for all $i \in \Gamma \neq \emptyset$. Since

$$\mathcal{K}_{R_T^{r_i*}}(A) = \bigwedge_{x\in X}(A(x) \to (R_T^{r_i}(x,-))^*) \in (\tau_T^{r_i})^*$$

$\mathbf{T}^*(B) = \mathbf{T}^*(\mathcal{K}_{R_T^{r_i*}}(A)) \geq r_i$, then $\mathbf{T}^*(B) \geq \bigvee_{i\in\Gamma} r_i = s$;i.e. $B \in (\tau_T^s)^* = \tau_{\mathcal{K}_{R^{s*}_{T^*}}}$;i.e. $B^* \in \tau_T^{s*} = \tau_{\mathcal{K}_{R_T^{s*}}}$. Since $\mathcal{K}_{R_T^s}(B^*) = B = \mathcal{K}_{R_T^{r_i*}}(A) \leq A^*$, $A \leq \mathcal{K}^*_{R_T^s}(B^*) = B^*$. Thus

$$\mathcal{K}_{R_T^{s*}}(A) \geq \mathcal{K}_{R_T^{s*}}(\mathcal{K}^*_{R_T^{s*}}(B^*)) = \mathcal{K}_{R_T^{s*}}(B^*) = B.$$

Since $s \geq r_i$, $\mathcal{K}_{R_T^{s*}}(A) \leq \mathcal{K}_{R_T^{r_i*}}(A) = B$. Thus $\mathcal{K}_{R_T^{s*}}(A) = B$.

(6) For each $A \in L^X$ with $A_i \leq A^*$, $\mathbf{T}(A_i) \geq r$, since $A_i \in \tau_T^r = \tau_{\mathcal{K}_{R_T^{r*}}}$ from Theorem 3.2(3), then

$$\mathcal{K}_{R_T^{r*}}(\bigvee_i A_i) = \bigwedge_i \mathcal{K}_{R_T^{r*}}(A_i) = \bigwedge_i A_i^*.$$

Since $\bigvee_i A_i \in \tau_{\mathcal{K}_{R_T^{r*}}} = \tau_T^r$ iff $(\bigvee_i A_i)^* \in \tau_{\mathcal{K}_{R^{r*}_{T^*}}} = \tau^r_{T^*}$, then

$$\mathcal{K}_{R^{r*}_{T^*}}((\bigvee_i A_i)^*) = \mathcal{K}_{R^{r*}_{T^*}}(\bigwedge_i A_i^*) = \bigvee_i A_i.$$

Since $\bigwedge_i A_i^* \geq A$. Thus

$$\mathcal{K}_{R^r_{T^*}}(A) \geq \mathcal{K}_{R^{r*}_{T^*}}(\bigwedge_i A_i^*) = \bigvee_i A_i = \bigvee\{A_i \mid A_i \leq A^*,\ \mathbf{T}(A_i) \geq r\}.$$

Since $\mathcal{K}_{R^{r*}_{T^*}}(\mathcal{K}^*_{R^{r*}_{T^*}}(A)) = \mathcal{K}_{R^{r*}_{T^*}}(A) \leq A^*$. Since

$$\mathcal{K}_{R^{r*}_{T^*}}(A) = \bigwedge_{x\in X}(A(x) \to (R^r_{T^*}(x,-))^*) \in \tau_T^r$$

So, $\bigvee\{A_i \mid A_i \leq A^*,\ \mathbf{T}(A_i) \geq r\} \geq \mathcal{K}_{R^{r*}_{T^*}}(A)$. Hence $\bigvee\{A_i \mid A_i \leq A,\ \mathbf{T}(A_i) \geq r\} = \mathcal{K}_{R^{r*}_{T^*}}(A)$ for all $A \in L^X$ and $r \in L$. Moreover, $\mathcal{K}_{R^{r*}_{T^*}}(\top_x)(y) = \bigwedge_{z\in X}(\top_x(z) \to R^{r*}_{T^*}(z,y)) = R^{r*}_{T^*}(x,y) = R_T^{-r*}(x,y)$.

(5) and (6) are similarly proved as (4) and (7), respectively. □

THEOREM 3.4. *Let* $\mathbf{T}$ *be an Alexandrov fuzzy topology on* X. *Then the following properties hold.*

(1) *Define* $\mathbf{T}_{K_T}: L^X \to L$ *as*

$$\mathbf{T}_{K_T}(A) = \bigvee\{r_i \in L \mid \mathcal{K}_{R_T^{r_i*}}(A) = A^*\}.$$

Then $\mathbf{T}_{K_T}$ *is an Alexandrov fuzzy topology on* X *such that* $\mathbf{T}_{K_T} = \mathbf{T}$.

(2) *Define* $\mathbf{T}_{K_{T^*}}: L^X \to L$ *as*

$$\mathbf{T}_{K_{T^*}}(A) = \bigvee\{r_i \in L \mid \mathcal{K}_{R_T^{-r_i*}}(A) = A^*\} = \bigvee\{r_i \in L \mid \mathcal{K}_{R_{T^*}^{r_i*}}(A) = A^*\}.$$

Then $\mathbf{T}_{K_{T^*}}$ *is an Alexandrov fuzzy topology on* X *such that* $\mathbf{T}_{K_{T^*}} = \mathbf{T}^*$.

(3) *There exists an Alexandrov fuzzy topology* $\mathbf{T}_K^r$ *such that*

$$\mathbf{T}_K^r(A) = e_{L^X}(A^*, \mathcal{K}_{R_T^{r*}}(A)).$$

If $r \le s$, *then* $\mathbf{T}_K^s \le \mathbf{T}_K^r$ *for all* $A \in L^X$.

(4) *There exists an Alexandrov fuzzy topology* $\mathbf{T}_K^{*r}$ *such that*

$$\mathbf{T}_K^{*r}(A) = e_{L^X}(A^*, \mathcal{K}_{R_T^{-r}}(A)).$$

If $r \le s$, *then* $\mathbf{T}_K^{*r} \le \mathbf{T}_K^{*s}$ *for all* $A \in L^X$.

(5) *Define* $\mathbf{T}_K: L^X \to L$ *as*

$$\mathbf{T}_K(A) = \bigvee\{r^* \in L \mid \mathbf{T}_K^r(A) = \top\}.$$

Then $\mathbf{T}_K = \mathbf{T} = \mathbf{T}_{K_T}$ *is an Alexandrov fuzzy topology on* X.

(6) *Define* $\mathbf{T}_{K^*}: L^X \to L$ *as*

$$\mathbf{T}_{K^*}(A) = \bigvee\{r^* \in L \mid \mathbf{T}_K^{*r}(A) = \top\}.$$

Then $\mathbf{T}_{K^*} = \mathbf{T}^* = \mathbf{T}_{K_{T^*}}$ *is an Alexandrov fuzzy topology on* X.

Proof. (1) We will show that $\mathbf{T}_{K_T} = \mathbf{T}$. Let $\mathcal{K}_{R_T^{r_i*}}(A) = A^*$. Since $\mathcal{K}_{R_T^{r_i*}}(A) \in (\tau_T^{r_i})^*$ and $\mathbf{T}(A) = \mathbf{T}^*(A^*) = \mathbf{T}^*(\mathcal{K}_{R_T^{r_i*}}(A)) \ge r_i$, then

$$\mathbf{T}_{K_T}(A) = \bigvee\{r_i \in L \mid \mathcal{K}_{R_T^{r_i}}(A) = A^*\} \le \mathbf{T}(A).$$

Since $\mathbf{T}(A) \ge \mathbf{T}(A)$ and $\tau_T^s = \tau_{\mathcal{K}_{R_T^{s*}}}$, then $\mathcal{K}_{R_T^{s*}}(A) = A$ where $\mathbf{T}(A) = s$. Thus

$$\mathbf{T}_{K_T}(A) = \bigvee\{r_i \in L \mid \mathcal{K}_{R_T^{r_i*}}(A) = A^*\} \ge \mathbf{T}(A).$$

Hence $\mathbf{T}_{K_T} = \mathbf{T}$.

(3) (T1) By Lemma 2.4(12), since $\alpha^* \odot R^r_T(z,x) \leq \alpha^*$,

$$\begin{aligned} \mathbf{T}^r_K(\alpha_X) &= \bigwedge_x(\alpha^*_X \to \mathcal{K}_{R^{r*}_T}(\alpha_X)(x)) \\ &= \bigwedge_x(\alpha^* \to \bigwedge_{z\in X}(\alpha \to R^{r*}_T(z,x))) \\ &= \bigwedge_x(\alpha^* \to \bigwedge_{z\in X}(R^r_T(z,x) \to \alpha^*)) \\ &= \bigwedge_x \bigwedge_{z\in X}(\alpha^* \odot R^r_T(z,x) \to \alpha^*) = \top. \end{aligned}$$

(T2)Since $\mathcal{K}_{R^{r*}_T}(\bigvee_{i\in\Gamma} A_i) = \bigwedge_{i\in\Gamma} \mathcal{K}_{R^{r*}_T}(A_i)$, by Lemma 2.4(8),

$$\begin{aligned} &\mathbf{T}^r_K(\bigvee_{i\in\Gamma} A_i) = e_{L^X}((\bigvee_{i\in\Gamma} A_i)^*, \mathcal{K}_{R^{r*}_T}(\bigvee_{i\in\Gamma} A_i)) \\ &= e_{L^X}(\bigwedge_{i\in\Gamma} A^*_i, \bigwedge_{i\in\Gamma} \mathcal{K}_{R^{r*}_T}(A_i)) \\ &\geq \bigwedge_{i\in\Gamma} e_{L^X}(A^*_i, \mathcal{K}_{R^{r*}_T}(A_i)) = \bigwedge_{i\in\Gamma} \mathbf{T}^r_K(A_i) \end{aligned}$$

Since $\mathcal{K}_{R^{r*}_T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigvee_{i\in\Gamma} \mathcal{K}_{R^{r*}_T}(A_i)$, by Lemma 2.4(8), we have

$$\begin{aligned} &\mathbf{T}^r_K(\bigwedge_{i\in\Gamma} A_i) = e_{L^X}((\bigwedge_{i\in\Gamma} A_i)^*, \mathcal{K}_{R^{r*}_T}(\bigwedge_{i\in\Gamma} A_i)) \\ &\geq e_{L^X}(\bigvee_{i\in\Gamma} A^*_i, \bigvee_{i\in\Gamma} \mathcal{K}_{R^{r*}_T}(A_i)) \\ &\geq \bigwedge_{i\in\Gamma} e_{L^X}(A^*_i, \mathcal{K}_{R^{r*}_T}(A_i)) = \bigwedge_{i\in\Gamma} \mathbf{T}^r_K(A_i) \end{aligned}$$

(T3) Since

$$\begin{aligned} &\alpha \to \mathcal{K}_{R^{r*}_T}(\alpha \odot A) = \mathcal{K}_{R^{r*}_T}(\alpha \to (\alpha \odot A)) \geq \mathcal{K}_{R^{r*}_T}(A) \\ &\text{iff } \mathcal{K}_{R^{r*}_T}(\alpha \odot A) \geq \alpha \odot \mathcal{K}_{R^{r*}_T}(A), \end{aligned}$$

by Lemma 2.4(8),

$$\begin{aligned} &\mathbf{T}^r_K(\alpha \odot A) = e_{L^X}((\alpha \odot A)^*, \mathcal{K}_{R^{r*}_T}(\alpha \odot A)) \\ &\geq e_{L^X}(\alpha \to A^*, \alpha \to \mathcal{K}_{R^{r*}_T}(A)) \\ &\geq e_{L^X}(A^*, \mathcal{K}_{R^{r*}_T}(A)) = \mathbf{T}^{r*}_K(A).\text{(by Lemma 2.4(8))} \end{aligned}$$

(T4)

$$\begin{aligned} &\alpha \to \mathcal{K}_{R^{r*}_T}(\alpha \to A) = \mathcal{K}_{R^{r*}_T}(\alpha \odot (\alpha \to A)) \geq \mathcal{K}_{R^{r*}_T}(A) \\ &\text{iff } \mathcal{K}_{R^{r*}_T}(\alpha \to A) \geq \alpha \odot \mathcal{K}_{R^{r*}_T}(A), \end{aligned}$$

by Lemma 2.4(8),

$$\begin{aligned} &\mathbf{T}^r_K(\alpha \to A) = e_{L^X}((\alpha \to A)^*, \mathcal{K}_{R^{r*}_T}(\alpha \to A)) \\ &= e_{L^X}(\alpha \odot A^*, \alpha \odot \mathcal{K}_{R^{r*}_T}(A)) \\ &\geq e_{L^X}(A^*, \mathcal{K}_{R^{r*}_T}(A)) = \mathbf{T}^r_K(A).\text{(by Lemma 2.4(10))} \end{aligned}$$

Hence $\mathbf{T}^r_K$ is an Alexandrov fuzzy topology. Since $\mathcal{K}_{R^{s*}_T} \leq \mathcal{K}_{R^{r*}_T}$ for $r \leq s$, $\mathbf{T}^s_K(A) = e_{L^X}(A, \mathcal{K}_{R^{s*}_T}(A)) \leq e_{L^X}(A, \mathcal{K}_{R^{r*}_T}(A)) = \mathbf{T}^r_K(A)$.

(5) Since $\mathbf{T}^r_K(A) = e_{L^X}(A^*, \mathcal{K}_{R^{r*}_T}(A)) = \top$ iff $A^* = \mathcal{K}_{R^{r*}_T}(A)$, by (1),

$$\begin{array}{ll} \mathbf{T}_K(A) & = \bigvee\{r \in L \mid \mathbf{T}^r_K(A) = \top\} \\ & = \bigvee\{r \in L \mid \mathcal{K}_{R^{r*}_T}(A) = A^*\} \\ & = \mathbf{T}_{K_T}(A) = \mathbf{T}(A). \end{array}$$

(2), (4) and (6) are similarly proved. □

Example 3.5. Let $(L = [0,1], \odot, \rightarrow, ^*)$ be a complete residuated lattice with a strong negation.

(1) Let $X = \{x, y, z\}$ be a set. Define a map $\mathbf{T} : [0,1]^X \rightarrow [0,1]$ as

$$\mathbf{T}(A) = A(x) \rightarrow A(z).$$

Trivially, $\mathbf{T}(\alpha_X) = 1$

Since $\alpha \odot A(x) \rightarrow \alpha \odot A(z) \geq A(x) \rightarrow A(z)$ from Lemma 2.4 (14), $\mathbf{T}(\alpha \odot A) \geq \mathbf{T}(A)$. Since $(\alpha \rightarrow A(x)) \rightarrow (\alpha \rightarrow A(z)) \geq A(x) \rightarrow A(z)$ from Lemma 2.4 (10), $\mathbf{T}(\alpha \rightarrow A) \geq \mathbf{T}(A)$. By Lemma 2.4 (8), $\mathbf{T}(\bigvee_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$ and $\mathbf{T}(\bigwedge_{i\in\Gamma} A_i) \geq \bigwedge_{i\in\Gamma} \mathbf{T}(A_i)$. Hence $\mathbf{T}$ is an Alexandrov fuzzy topology.

Since $\mathbf{T}(A) = A(x) \rightarrow A(z) \geq r$, then $A(z) \geq A(x) \odot r$. Put $A(x) = 1, A(y) = 0$. So, $R^r_T(x,y) = \bigwedge\{A(x) \rightarrow A(y) \mid \mathbf{T}(A) \geq r\} = 0$ and $R^r_T(x,z) = \bigwedge\{A(x) \rightarrow A(z) \mid \mathbf{T}(A) \geq r\} = r$

$$\begin{pmatrix} R^r_T(x,x) = 1 & R^r_T(x,y) = 0 & R^r_T(x,z) = r \\ R^r_T(y,x) = 0 & R^r_T(y,y) = 1 & R^r_T(y,z) = 0 \\ R^r_T(z,x) = 0 & R^r_T(z,y) = 0 & R^r_T(z,z) = 1 \end{pmatrix}$$

By Theorem 3.1(3), we obtain $\mathcal{K}_{R^{r*}_T}(A)(y) = \bigwedge_{x\in X}(A(x) \rightarrow R^{r*}_T(x,y))$ such that

$$\begin{array}{ll} \mathcal{K}_{R^{r*}_T}(A) & = (A(x) \rightarrow 0, A(y) \rightarrow 0, (A(x) \rightarrow r^*) \wedge (A(z) \rightarrow 0)) \\ & = (A^*(x), A^*(y), (A(x) \rightarrow r^*) \wedge A^*(z))) \end{array}$$

If $A^*(z) \leq A(x) \rightarrow r^*$, then $\mathcal{K}_{R^{r*}_T}(A) = A^*$, that is, $A \in \tau_{\mathcal{K}_{R^{r*}_T}}$. If $\mathcal{K}_{R^{r*}_T}(A) = A^*$, then $(A(x) \rightarrow r^*) \wedge A^*(z) = A^*(z)$, that is, $A^*(z) \leq A(x) \rightarrow r^*$. Hence $A^*(z) \leq A(x) \rightarrow r^*$ iff $A^*(z) \leq (A(x) \odot r)^*$ iff $A(z) \geq A(x) \odot r$ iff $r \leq (A(x) \rightarrow A(z)) = \mathbf{T}(A)$ iff $A \in \tau_{\mathcal{K}_{R^{r*}_T}}$.

$$\begin{array}{ll} \mathbf{T}_{K_T}(A) & = \bigvee\{r \in L \mid \mathcal{K}_{R^{r*}_T}(A) = A^*\} \\ & = \bigvee\{r \in L \mid r \leq A(x) \rightarrow A(z)\} \\ & = A(x) \rightarrow A(z) = \mathbf{T}(A). \end{array}$$

Moreover,

$$\mathcal{K}_{R^{r*}_T}(A^*) = (A(x), A(y), (A^*(x) \rightarrow r^*) \wedge A(z))).$$

From Theorem 3.4(1), we obtain

$$\begin{array}{ll} \mathbf{T}^r_K(A) & = \bigwedge_{x\in X}(A^*(x) \to \mathcal{K}_{R^{r*}_T}(A)(x)) \\ & = A^*(z) \to (r \to A^*(x)) = r \to (A^*(z) \to A^*(x)). \end{array}$$

$$\begin{array}{ll} \mathbf{T}_K(A) & = \bigvee\{r \in L \mid \mathbf{T}^r_K(A) = 1\} \\ & = \bigvee\{r \in L \mid r \to (A^*(z) \to A^*(x)) = 1\} \\ & = A(x) \to A(z) = \mathbf{T}(A). \end{array}$$

Hence $\mathbf{T}_K = \mathbf{T}_{K_T} = \mathbf{T}$.

(2) By (1), we obtain a map $\mathbf{T}^* : [0,1]^X \to [0,1]$ as

$$\mathbf{T}^*(A) = A^*(x) \to A^*(z) = A(z) \to A(x).$$

Since $\mathbf{T}^*(A) = A(z) \to A(x) \geq r$, then $A(x) \geq A(z) \odot r$. Put $A(z) = 1, A(y) = 0$. So, $R^r_{T^*}(z,y) = \bigwedge\{A(z) \to A(y) \mid \mathbf{T}^*(A) \geq r\} = 0$ and $R^r_{T^*}(z,x) = \bigwedge\{A(z) \to A(x) \mid \mathbf{T}(A) \geq r\} = r$

$$\begin{pmatrix} R^r_{T^*}(x,x) = 1 & R^r_{T^*}(x,y) = 0 & R^r_{T^*}(x,z) = 0 \\ R^r_{T^*}(y,x) = 0 & R^r_{T^*}(y,y) = 1 & R^r_{T^*}(y,z) = 0 \\ R^r_{T^*}(z,x) = r & R^r_{T^*}(z,y) = 0 & R^r_{T^*}(z,z) = 1 \end{pmatrix}$$

Moreover, $R^r_{T^*}(x,y) = R^{-r}_T(x,y) = R^r_T(y,x)$ for all $x,y \in X$.

$$\mathcal{K}_{R^{r*}_{T^*}}(A)(y) = \bigwedge_{x\in X}(A(x) \to R^{r*}_{T^*}(x,y)).$$

$$\mathcal{K}_{R^r_{T^*}}(A) = (A^*(x) \wedge (A(z) \to r), A^*(y), A^*(z))$$

Then $A^*(x) \leq A(z) \to r$ iff $\mathcal{K}_{R^{r*}_{T^*}}(A) = A^*$. Moreover, since $\mathbf{T}^*(A) = A(z) \to A(x) \geq r$ iff $A(z) \odot r \leq A(x)$ iff $A^*(x) \leq A(z) \to r$, then $A \in \tau^r_{T^*}$ iff $A \in \tau_{\mathcal{K}_{R^r_{T^*}}}$. Thus $\tau^r_{T^*} = \tau_{\mathcal{K}_{R^{r*}_{T^*}}}$. Moreover,

$$\begin{array}{ll} \mathbf{T}_{K_{T^*}}(A) & = \bigvee\{r \in L \mid \mathcal{K}_{R^{r*}_{T^*}}(A) = A^*\} \\ & = A(z) \to A(x) = \mathbf{T}^*(A). \end{array}$$

Moreover, we obtain

$$\begin{array}{ll} \mathbf{T}^{*r}_K(A) & = \bigwedge_{x\in X}(A^*(x) \to \mathcal{K}_{R^{r*}_{T^*}}(A)(x)) \\ & = A^*(x) \to (A(z) \to r^*) = r \to (A(z) \to A(x)). \end{array}$$

$$\begin{array}{ll} \mathbf{T}_{K^*}(A) & = \bigvee\{r \in L \mid \mathbf{T}^{*r}_K(A) = 1\} \\ & = A(z) \to A(x) = \mathbf{T}^*(A). \end{array}$$

Hence $\mathbf{T}_{K^*} = \mathbf{T}_{K_{T^*}} = \mathbf{T}^*$.

$$\mathcal{K}_{R^{r*}_{T^*}}(1_x)(z) = \bigvee\{B(z) \mid B \leq 1^*_x,\ \mathbf{T}(B) \geq r\}$$

Since $B(x)=0$ and $\mathbf{T}(B)=0\to B(z)=1\geq r$, then $\mathcal{K}_{R^{r*}_{T^*}}(1_x)(z)=1$.

$$\mathcal{K}_{R^{r*}_{T^*}}(1_z)(x)=\bigvee\{B(x)\mid B\leq 1^*_z,\ \mathbf{T}(B)\geq r\}$$

Since $B(z)=0$ and $\mathbf{T}(B)=B(x)\to 0\geq r$, then $\mathcal{K}_{R^{r*}_{T^*}}(1_z)(x)=r^*$.

$$\begin{pmatrix} \mathcal{K}_{R^{r*}_{T^*}}(1_x)(x)=0 & \mathcal{K}_{R^{r*}_{T^*}}(1_x)(y)=1 & \mathcal{K}_{R^{r*}_{T^*}}(1_x)(z)=1 \\ \mathcal{K}_{R^{r*}_{T^*}}(1_y)(x)=1 & \mathcal{K}_{R^{r*}_{T^*}}(1_y)(y)=0 & \mathcal{K}_{R^{r*}_{T^*}}(1_y)(z)=1 \\ \mathcal{K}_{R^{r*}_{T^*}}(1_z)(x)=r^* & \mathcal{K}_{R^{r*}_{T^*}}(1_z)(y)=1 & \mathcal{K}_{R^{r*}_{T^*}}(1_z)(z)=0 \end{pmatrix}$$

Then $\mathcal{K}_{R^{r*}_{T^*}}(1_x)(y)=R^{r*}_{T^*}(x,y)$.

$$\mathcal{K}_{R^{r*}_{T}}(1_x)(z)=\bigvee\{B(z)\mid B\leq 1^*_x,\ \mathbf{T}^*(B)\geq r\}$$

Since $B(x)=0$ and $\mathbf{T}^*(B)=B(z)\to 0\geq r$, then $\mathcal{K}_{R^{r*}_{T}}(1_x)(z)=r^*$.

$$\mathcal{K}_{R^{r*}_{T}}(1_z)(x)=\bigvee\{B(x)\mid B\leq 1^*_z,\ \mathbf{T}^*(B)\geq r\}$$

Since $B(z)=0$ and $\mathbf{T}^*(B)=0\to B(x)=1\geq r$, then $\mathcal{K}_{R^{r*}_{T}}(1^*_z)(x)=1$.

$$\begin{pmatrix} \mathcal{K}_{R^{r*}_{T}}(1_x)(x)=0 & \mathcal{K}_{R^{r*}_{T}}(1_x)(y)=1 & \mathcal{K}_{R^{r*}_{T}}(1_x)(z)=r^* \\ \mathcal{K}_{R^{r*}_{T}}(1_y)(x)=1 & \mathcal{K}_{R^{r*}_{T}}(1_y)(y)=0 & \mathcal{K}_{R^{r*}_{T}}(1_y)(z)=1 \\ \mathcal{K}_{R^{r*}_{T}}(1_z)(x)=1 & \mathcal{K}_{R^{r*}_{T}}(1_z)(y)=1 & \mathcal{K}_{R^{r*}_{T}}(1_z)(z)=0 \end{pmatrix}$$

Then $\mathcal{K}_{R^{r*}_{T}}(1_x)(y)=R^{r*}_{T}(x,y)$.

(3) Let $(L=[0,1],\odot,\to,^*)$ be a complete residuated lattice with a strong negation defined by, for each $n\in N$,

$$x\odot y=((x^n+y^n-1)\vee 0)^{\frac{1}{n}},\ \ x\to y=(1-x^n+y^n)^{\frac{1}{n}}\wedge 1,\ x^*=(1-x^n)^{\frac{1}{n}}.$$

By (1) and (2), we obtain

$$\mathbf{T}(A)=(1-A(x)^n+A(z)^n)^{\frac{1}{n}}\wedge 1,\ \ \mathbf{T}^*(A)=(1-A(z)^n+A(x)^n)^{\frac{1}{n}}\wedge 1.$$

$$R^{r*}_{T}=\begin{pmatrix} 1 & 0 & r \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\ \ R^{r*}_{T^*}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ r & 0 & 1 \end{pmatrix}$$

$$\mathcal{K}_{R^{r*}_{T}}(A)=(A^*(x),A^*(y),A^*(z)\wedge(1-r^n+(A^*(x))^n)^{\frac{1}{n}})$$

$$\mathcal{K}_{R^{r*}_{T^*}}(A)=(A^*(x)\wedge(1-r^n+(A^*(z))^n)^{\frac{1}{n}},A^*(y),A^*(z)).$$

Since $\mathbf{T}(A)=(1-A(x)^n+A(z)^n)^{\frac{1}{n}}\wedge 1\geq r$, we have

$$\begin{aligned} \tau^r_T=\tau_{\mathcal{K}_{R^{r*}_{T}}} &=\{A\in L^X\mid A^n(z)-A^n(x)\geq 1-r^n\} \\ \tau^r_{T^*}=\tau_{\mathcal{K}_{R^{r*}_{T^*}}} &=\{A\in L^X\mid A^n(x)-A^n(z)\geq 1-r^n\}. \end{aligned}$$

$$\begin{array}{ll}\mathbf{T}^r_K(A) & = r \to (A(x) \to A(z)) = (2 - r^n - A(x)^n + A(z)^n)^{\frac{1}{n}} \wedge 1 \\ \mathbf{T}^{*r}_K(A) & = r \to (A(z) \to A(x)) = (2 - r^n - A(z)^n + A(x)^n)^{\frac{1}{n}} \wedge 1.\end{array}$$

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