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SOME PROOFS OF THE CLASSICAL INTEGRAL HARDY INEQUALITY

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Abstract. We present some proofs of the classical integral Hardy inequality. Our approach makes use of continuous functions with compact support in $(0, \infty)$, homogeneity of the norm and Schur's criterion for integral operators.

1. Introduction

The classical integral inequality announced by G. H. Hardy in 1920 is given by

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{1}$$

where $p > 1$, $x > 0$, f is a nonnegative measurable function on $(0, \infty)$ and the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible [4]. This interesting result (1) was later proved by Hardy himself in 1925 (see [1], [5], [7], [8], [9] and the references therein.) Inequality (1) can also be written as

$$\int_0^\infty F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx, \tag{2}$$

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where $0 < F(x) = \frac{1}{x}\int_0^x f(t)dt < \infty$, $f > 0$.
The inequalities (1) and (2) are very popular in the research environment. See also [6].

Our task in this paper is mainly to deepen understanding of the Hardy inequality (2) by providing elaborate proofs.

2. Preliminary Notes

We define continuous functions and present some auxilliary results.

Definition (Continuous functions) [11]. Let X be a subset of the set of real numbers $\Re$, and let $f : X \to \Re$ be a function. Let $x_0 \in X$. We say that f is continuous at x_0 if and only if we have $\lim_{x\to x_0} f(x) = f(x_0)$ for every $x \in X$. In other words, the limit of $f(x)$ as x converges to x_0 in X exists and is equal to $f(x_0)$.

Support of a function. Let I be a nonempty open set in $\Re^n$, and let f be a continuous function on I. The support of f, denoted by supp(f), is defined to be the complement of the largest open set on which f is zero. That is

$$\text{supp(f)} = \overline{\{x \in I : f(x) \neq 0\}},$$

the closure of the set $x \in I$ where $f(x) \neq 0$. (See [3], p. 134).

Fatou's Lemma ([2], p. 52). Let X be a measure space with measure μ. Let $\{f_n\}$ be any sequence of measurable functions on X with range in $[0, \infty]$. For each positive integer n,

$$\int_X (\liminf_{n\to\infty} f_n)d\mu \leq \liminf_{n\to\infty} \int_X f_n d\mu. \tag{3}$$

Hölder's inequality ([2], p.182). Suppose $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$ (that is $p + q = pq$). Let X be a measure space with measure μ. If f and g are measurable functions on X with range in $[0, \infty]$, then

$$\int_X fg d\mu \leq \left\{\int_X f^p d\mu\right\}^{\frac{1}{p}} \left\{\int_X g^q d\mu\right\}^{\frac{1}{q}}, \tag{4}$$

In particular, if $f \in L^p$ and $g \in L^q$, then $fg \in L^1$, and in this case equality holds in (4) if and only if $\alpha|f|^p = \beta|g|^q$ almost everywhere for some constants α, β with $\alpha\beta \neq 0$. See [2] and also [10] for proofs of inequalities (3) and (4).

Proposition 1. ([2], p.195). *For $f \in L^p$ and $g \in L^q$, let*

$$Tf(x) = x^{-1}\int_0^x f(y)dy, \quad Sg(y) = \int_y^{+\infty} x^{-1}g(x)dx.$$

Then for $1 < p \leq \infty$ and $1 \leq q < \infty$,

$$\|Tf\|_p \leq \frac{p}{p-1}\|f\|_p, \quad \|Sg\|_q \leq q\|g\|_q.$$

Proof. Let

$$K(x,y) = \begin{cases} \frac{1}{x} & \text{if} \quad 0 < y < x \\ 0 & \text{otherwise} \end{cases}.$$

Then

$$\int_0^{+\infty} |K(1,y)|y^{-\frac{1}{p}}dy = \int_0^1 y^{-\frac{1}{p}}dy = \frac{p}{p-1} = q,$$

where $\frac{1}{p} + \frac{1}{q} = 1$, yielding the result. □

We now present our main results which are basically the different approaches to the proof of inequality (2). We denote by $C_c(0,+\infty)$, the set of all continuous functions with compact support in $(0,+\infty)$.

3. First Proof

Integration by parts and Hölder's inequality are essentially applied here.

3.1. Case 1: Let $p > 1$, f is positive and continuous with compact support in $(0,+\infty)$ and F is positive and differentiable on $[0,+\infty)$.

Setting $u = F^p$ and $dv = dx$ implies $du = pF^{p-1}F'dx$ and $v = x$.

Consider (a, A_0) with $0 < a < A_0 < \infty$ so that the supp$f \subset [a, A_0]$. Integration by parts of $\int_a^{A_0} F^p dx$ gives

$$\int_a^{A_0} F^p dx = [xF^p]_a^{A_0} - p\int_a^{A_0} xF^{p-1}F' dx. \tag{5}$$

But $\int_0^A F^p dx = \int_0^a F^p dx + \int_a^{A_0} F^p dx + \int_{A_0}^A F^p dx$ for $0 < A_0 < A < \infty$. Since $f \in C_c(0, +\infty)$ with supp$f \subset [a, A_0]$ then $F(x) = 0$ for $0 \leq x < a$ and $A_0 < x \leq A$. Thus

$$\int_0^{A_0} F^p(x)dx = \int_0^A F^p(x)dx$$

and (5) becomes

$$\int_0^A F^p(x)dx = AF^p(A) - p\int_0^A xF^{p-1}(x)F'(x)dx. \tag{6}$$

But $F(x) = \frac{1}{x}\int_0^x f(t)dt$ implies $xF(x) = \int_0^x f(t)dt$. Differentiating gives $xF'(x) + F(x) = f(x)$
Thus (6) becomes

$$\begin{aligned}\int_0^A F^p(x)dx &= AF^p(A) - p\int_0^A F^{p-1}(x)\{f(x) - F(x)\}dx\\ &= AF^p(A) - p\int_0^A F^{p-1}(x)f(x)dx + p\int_0^A F^p(x)dx,\end{aligned}$$

$$\int_0^A F^p(x)dx = \frac{AF^p(A)}{1-p} + \frac{p}{p-1}\int_0^A F^{p-1}(x)f(x)dx. \tag{7}$$

By Hölder's inequality,

$$\int_0^A F^{p-1}(x)f(x)dx \leq \left(\int_0^A F^{(p-1)q}(x)dx\right)^{\frac{1}{q}}\left(\int_0^A f^p(x)dx\right)^{\frac{1}{p}} \tag{8}$$

where $\frac{1}{p} + \frac{1}{q} = 1$ or $q = \frac{p}{p-1}$.
Putting (8) into (7), then
(9)

$$\int_0^A F^p(x)dx \leq \frac{AF^p(A)}{1-p} + \frac{p}{p-1}\left(\int_0^A F^{(p-1)q}(x)dx\right)^{\frac{1}{q}}\left(\int_0^A f^p(x)dx\right)^{\frac{1}{p}}.$$

Let $I = \int_0^A f(t)dt$ so that $F(A) = \frac{I}{A}$. Then $AF^p(A) = \frac{I^p}{A^{p-1}}$ turns to 0 as $A \to \infty$. Thus (9) simplies to

$$\int_0^\infty F^p(x)dx \leq \frac{p}{p-1}\left(\int_0^\infty F^p(x)dx\right)^{(1-\frac{1}{p})}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}. \tag{10}$$

Since $\left(\int_0^\infty F^p(x)dx\right)^{(1-\frac{1}{p})} > 0$, then

$$\left(\int_0^\infty F^p(x)dx\right)^{\frac{1}{p}} \leq \frac{p}{p-1}\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}},$$

Thus

$$\int_0^\infty F^p(x)dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(x)dx. \quad \square$$

3.2. Case 2: We consider f not necessarily positive. Let $f \in L^p(0,+\infty)$ and set $f_+ = \max(f,0)$ and $f_- = -\min(0,f)$. Consider sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $C_c(0,+\infty)$ such that $f_n \longrightarrow f$ in $L^p(0,+\infty)$. For a fixed $x > 0$, $F(x) = \frac{1}{x}\int_0^x f(t)dt$ is well defined for $f \geq 0$. By Hölder's inequality,

$$\begin{aligned}
\left|\frac{1}{x}\int_0^x f(t)dt\right| &\leq \frac{1}{x}\int_0^x |f(t)|dt \\
&\leq \frac{1}{x}\left(\int_0^x |f(t)|^p dt\right)^{\frac{1}{p}}\left(\int_0^x 1^q dt\right)^{\frac{1}{q}} \\
&\leq x^{-\frac{1}{p}}\left(\int_0^x |f(t)|^p dt\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence, by Fubini, we get

$$\int_0^\infty |F(x)|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |(f(x)|^p dx,$$

where the constant $\left(\frac{p}{p-1}\right)^p$ is the best possible.
Also set $f = f_+ - f_-$, $|f| = f_+ + f_-$ and $F = F_+ - F_-$, $|F| = F_+ + F_-$

and $\int |F| < \infty$. Let $\|f_n - f\| < \frac{1}{n}$. Then

$$\begin{aligned}
|F_n(x) - F(x)| &= \left|\frac{1}{x}\int_0^x f_n(t)dt - \frac{1}{x}\int_0^x f(t)dt\right| \\
&\leq \frac{1}{x}\int_0^x |f_n(t) - f(t)|\, dt \\
&\leq \frac{1}{x}\left(\int_0^x |f_n(t) - f(t)|^p dt\right)^{\frac{1}{p}} \left(\int_0^x 1^q dt\right)^{\frac{1}{q}} \quad by\ (4) \\
&\leq x^{-\frac{1}{p}} \|f_n - f\|_p \\
&< \frac{x^{-\frac{1}{p}}}{n} \to 0 \quad as \quad n \to \infty.
\end{aligned}$$

This shows pointwise convergence: $|F_n(x)| \to |F(x)|$ as $\to \infty$. Therefore

$$\int_0^\infty |F_n(x)|^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty |f_n(x)|^p dx. \tag{11}$$

Suppose that f is positive and so is F. Suppose also that f_n is positive which implies F_n is positive. Then

$$\begin{aligned}
\int_0^{+\infty} |F(x)|^p dx &= \int_0^{+\infty} \lim_{n\to+\infty} |F_n(x)|^p dx \\
&\leq \lim_{n\to+\infty} \int_0^{+\infty} |F_n(x)|^p dx \quad \text{(Fatou's lemma)} \\
&\leq \lim_{n\to+\infty} \left[\left(\frac{p}{p-1}\right)^p \int_0^{+\infty} |f_n(x)|^p dx\right] \\
&\leq \left(\frac{p}{p-1}\right)^p \int_0^{+\infty} |f(x)|^p dx. \quad \square
\end{aligned}$$

4. Second Proof

The approach here makes use of homogeneity of a norm and the use of a kernel.

4.1. Case 1: Homogeneity of a norm.
Let

$$F(x) = \frac{1}{x}\int_0^x f(t)dt = \int_0^1 f(tx)dt.$$

Set $f_t(x) = f(tx)$. By Minkowski inequality for integrals,

$$\begin{aligned} \|F(x)\|_p &\leq \int_0^1 \|f_t(x)\|_p dt \\ &\leq \int_0^1 \left(\int_0^1 |f_t(x)|^p dx \right)^{\frac{1}{p}} dt \end{aligned}$$

By change of variables with $s = tx$, $ds = tdx$, and also by Fubini's theorem we have

$$\begin{aligned} \|F(x)\|_p &\leq \int_0^1 \left(\int_0^1 |f(s)|^p \frac{ds}{t} \right)^{\frac{1}{p}} dt \\ &\leq \int_0^1 t^{-\frac{1}{p}} dt \left(\int_0^1 |f(s)|^p ds \right)^{\frac{1}{p}} \end{aligned}$$

$$\|F(x)\|_p \leq \frac{p}{p-1} \|f(s)\|_p$$

Hence

$$\|F(x)\|_p^p \leq \left(\frac{p}{p-1} \right)^p \|f\|_p^p. \quad \square$$

4.2. Case 2: Kernel's approach.

Let $K > 0$ be a Lebesgue measurable function on $(0, \infty)$ x $(0, \infty)$ and define

$$K(x, y) = \begin{cases} \frac{1}{x} & \text{if} \quad 0 < y < x \\ 0 & \text{otherwise} \end{cases} .$$

Let $f \in L^p$ and consider the integral operator

$$Tf(x) = \int_0^\infty |K(x, y) f(y)| dy.$$

By change of variables with $y = xz$, $dy = xdz$, we have

$$Tf(x) = \int_0^\infty |K(x, xz)| f(xz) x dz.$$

Set $f_z(x) = f(xz)$ and by Minkowski inequality for integrals, we have

$$\|Tf(x)\|_p \leq \int_0^\infty |K(1,z)| \|f_z(x)\|_p dz$$
$$\leq \int_0^1 \left(\int_0^\infty |f_z(x)|^p dx \right)^{\frac{1}{p}} dz$$

Again by change of variables with $xz = y$, $zdx = dy$ and by Fubini's theorem, we have

$$\|Tf(x)\|_p \leq \int_0^1 \left(\int_0^\infty |f(y)|^p \frac{dy}{z} \right)^{\frac{1}{p}} dz$$
$$\leq \int_0^1 z^{-\frac{1}{p}} dz \left(\int_0^\infty |f(y)|^p dy \right)^{\frac{1}{p}}$$
$$\leq \left(\frac{p}{p-1} \right) \|f(y)\|_p$$

Hence

$$\|Tf\|_p^p \leq \left(\frac{p}{p-1} \right)^p \|f\|_p^p. \qquad \square$$

Remark 1. Let us remark that the proofs for the cases 1 and 2 here can be described as almost the same, except that the mention of a kernel and its application is demonstrated in case 2 .

5. Third Proof

Let K be a Lebesgue measurable function on $(0, \infty)$ x $(0, \infty)$. For $1 < p < \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$, define a nonnegative kernel

$$K(x,y) = \begin{cases} \frac{1}{x} & \text{if} \quad 0 < y < x \\ 0 & \text{otherwise} \end{cases} .$$

and consider the integral operator

$$(12) \qquad Tf(x) = \int_0^{+\infty} K(x,y) f(y) dy = \frac{1}{x} \int_0^x f(y) dy = F(x)$$

Let $h(t) = t^\alpha$ for $-1 < \alpha < 0$. By Proposition 1, we compute

$$\int_0^{+\infty} K(x,y)h(y)dy = \int_0^x \frac{1}{x} y^\alpha dy = \frac{1}{x}\frac{x^{(\alpha+1)}}{\alpha+1} = \frac{x^\alpha}{\alpha+1} \tag{13}$$

and

$$\int_0^{+\infty} K(x,y)h(x)dx = \int_y^{+\infty} \frac{1}{x} x^\alpha dx = \int_y^{+\infty} x^{(\alpha-1)} dx = -\frac{1}{\alpha} y^\alpha. \tag{14}$$

Introduce y^α into (12) and applying Hölders inequality, we have

$$\begin{aligned}
Tf(x) &= \int_0^{+\infty} K(x,y) y^\alpha \frac{f(y)}{y^\alpha} dy \\
&\le \left(\int_0^{+\infty} K(x,y) y^{q\alpha} dy\right)^{\frac{1}{q}} \left(\int_0^{+\infty} K(x,y)\left(\frac{|f(y)|}{y^\alpha}\right)^p dy\right)^{\frac{1}{p}} \\
&\le \left(\frac{x^{q\alpha}}{q\alpha+1}\right)^{\frac{1}{q}} \left(\int_0^{+\infty} K(x,y)\left(\frac{|f(y)|}{y^\alpha}\right)^p dy\right)^{\frac{1}{p}} \quad \{\text{by (13)}\} \\
&\le \left(\frac{1}{q\alpha+1}\right)^{\frac{1}{q}} x^\alpha \left(\int_0^{+\infty} K(x,y)\left(\frac{|f(y)|}{y^\alpha}\right)^p dy\right)^{\frac{1}{p}} \qquad (15)
\end{aligned}$$

Let $\lambda = \left(\frac{1}{q\alpha+1}\right)^{\frac{1}{q}}$. Thus (15) becomes

$$|Tf(x)|^p \le \lambda^p x^{p\alpha} \left(\int_0^{+\infty} K(x,y)\left(\frac{|f(y)|}{y^\alpha}\right)^p dy\right) \tag{16}$$

By Fubini's theorem, we get

$$\begin{aligned}
\int_0^{+\infty} |Tf(x)|^p dx &\le \lambda^p \int_0^{+\infty} \left(x^{p\alpha}\left(\int_0^{+\infty} K(x,y)\left(\frac{|f(y)|}{y^\alpha}\right)^p dy\right)\right) dx \\
&\le \lambda^p \int_0^{+\infty} \left(\int_0^{+\infty} K(x,y) x^{p\alpha} dx\right)\left(\frac{|f(y)|}{y^\alpha}\right)^p dy. \qquad (17)
\end{aligned}$$

From (14), $Th(y) = \int_0^{+\infty} K(x,y)x^{p\alpha}dx = -\frac{y^{p\alpha}}{p\alpha}$. Thus (17) becomes

$$\begin{aligned}\int_0^{+\infty} |Tf(x)|^p dx &\leq \lambda^p \int_0^{+\infty} -\frac{y^{p\alpha}}{p\alpha}\frac{|f(y)|^p}{y^{p\alpha}}dy\\ &\leq -\frac{\lambda^p}{p\alpha}\int_0^{+\infty} |f(y)|^p dy\\ &\leq C\int_0^{+\infty} |f(y)|^p dy\end{aligned}$$

where $C = -\frac{\lambda^p}{p\alpha}$ for $-1 < \alpha < 0$. Equivalently

$$\|Tf(x)\|_P^p \leq C\|f(y)\|_p^p. \qquad \square$$

REMARK 2. The Schur criterion discussed above shows that the operator T is bounded on $L^p(0, +\infty)$ with $\|T\| \leq C$. See ([12], p. 45) for discussions on Schur test.

6. Conclusion

Precise proofs for the classical integral Hardy inequality were presented. A number of useful applications of some important theorems such as Fatou's lemma and Fubini's theorem as well as Hölder and Minkowski inequalities were provided.

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