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# QUASI-ELLIS GROUPS AND SOME SUBGROUPS OF THE AUTOMORPHISM GROUP

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Abstract. In this paper we give some relationships between quasi-Ellis groups and some subgroups of the automorphism group. In particular, we investigate several characterizations on some subgroups of the automorphism group.

## 1. Introduction

Universal minimal sets were studied by R. Ellis in [3]. S. Glasner introduced the Ellis group which is a certain group of the universal minimal set In [5]. Given a homomorphism of pointed minimal sets $\pi : (X, x_0) \to (Y, y_0)$, we can define quasi-Ellis groups $\mathcal{S}(X, x_0)$ and $\mathcal{S}(Y, y_0)$ which are generalizations of the Ellis groups and give some relationships between the homomorphism and quasi-Ellis groups.

Let $G$ be the automorphism group of universal minimal set $M$. Given a minimal set $X$ and a homomorphism $\gamma : M \to X$, we may define subgroups $G(X, \gamma)$ and $S(X, \gamma)$ of $G$, and study some characterizations on $G(X, \gamma)$ and $S(X, \gamma)$. In particular, we investigate $G$ and the subgroup of $M$ are isomorphic.

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## 2. Preliminaries

A *transformation group*, or *flow*, $(X, T)$, will consist of a jointly continuous action of the topological group $T$ on the compact Hausdorff space $X$. The group $T$, with identity $e$, is assumed to be topologically discrete and remain fixed throughout this paper, so we may write $X$ instead of $(X, T)$.

A *point transitive flow*, $(X, x_0)$ consists of a flow $X$ with a distinguished point $x_0$ which has dense orbit.

A *homomorphism* of flows is a continuous, equivariant map. A homomorphism whose domain is point transitive is determined by its value at a single point. A one-one homomorphism of $X$ onto $X$ is called an automorphism of $X$. We denote the group of automorphisms of $X$ by $A(X)$.

A flow is said to be *minimal* if every point has dense orbit. Minimal flows are also referred to as minimal sets. $M$ is said to be a *universal minimal set* if it is a minimal set such that every minimal set is a homomorphic image of $M$. A homomorphism whose range is minimal is always surjective.

Given a flow $(X, T)$, we may regard $T$ as a set of self-homeomorphisms of $X$. We define $E(X)$, the *enveloping semigroup* of $X$ to be the closure of $T$ in $X^X$, taken with the product topology. $E(X)$ is at once a flow and a sub-semigroup of $X^X$. The minimal right ideals of $E(X)$, considered as a semigroup, coincide with the minimal sets of $E(X)$.

The points $x, x' \in X$ are said to be *proximal* if there exists a net $(t_i)$ in $T$ such that $\lim xt_i = \lim x't_i$. The points $x, x' \in X$ are said to be *distal* if either $x = x'$ or $x$ and $x'$ are not proximal. Thus if $x$ and $x'$ are both proximal and distal, they must be equal. The set of all proximal pairs in $X$ will be denoted $P(X, T)$ or simply $P(X)$. $X$ is said to be *distal* if $P(X) = \triangle_X$, the diagonal of $X \times X$ and is said to be *proximal* if $P(X) = X \times X$. Given any point $x \in X$, we define $P(x) = \{x' \in X \mid (x, x') \in P(X)\}$.

A homomorphism $\pi : X \to Y$ is said to be *proximal* (resp. *distal*) if whenever $x, x' \in \pi^{-1}(y)$ then $x$ and $x'$ are proximal (resp. distal).

A homomorphism $\pi : X \to Y$ is said to be *regular* if whenever $x, x' \in X$ with $\pi(x) = \pi(x')$, then $(\phi(x), x') \in P(X)$ for some $\phi \in A(X)$.

If $E$ is some enveloping semigroup, and there exists a homomorphism $\theta : (E, e) \to (E(X), e)$ we say that $E$ is an *enveloping semigroup for $X$*.

If such a homomorphism exists, it must be unique, and, given $x \in X$ and $p \in E$ we may write $xp$ to mean $x\theta(p)$ unambiguously.

LEMMA 2.1. ([6]) *Let $E$ be an enveloping semigroup for $X$ and let $I$ be a minimal right ideal in $E$. The following are true :*

(1) *The set $J(I)$ of idempotent elements in $I$ is non-empty.*

(2) *$pI = I$ for all $p \in I$.*

(3) *$up = p$ whenever $p \in I$ and $u \in J(I)$.*

(4) *If $u \in J(I)$ then $Iu$ is a group with identity $u$.*

(5) *If $p \in I$ then there exists a unique $u \in J(I)$ with $pu = p$.*

(6) *Given $x \in X$, the following are equivalent :*

(a) *$x$ is an almost periodic point.*

(b) *$\overline{xT} = xI$.*

(c) *$x = xu$ for some $u \in J(I)$.*

LEMMA 2.2. ([6]) *Let $E$ be an enveloping semigroup for $X$, Then for any points $x, x' \in X$, (a) and (b) are equivalent :*

(a) *$(x, x') \in P(X, T)$.*

(b) *There exists a minimal right ideal $I$ in $E$ such that $xp = x'p$ for every $p \in I$.*

*Moreover, if $X$ is minimal, (a) and (b) are equivalent to :*

(c) *There exists $u \in J(I)$ such that $x' = xu$.*

LEMMA 2.3. ([6]) *If $(X, x)$ and $(Y, y)$ are point transitive flows, and $E$ is an enveloping semigroup for both $X$ and $Y$, there exists a unique homomorphism $\psi : (X, x) \to (Y, y)$ if and only if $xp = xq$ for $p, q \in E$ implies $yp = yq$.*

LEMMA 2.4. ([3]) *The following are equivalent :*

(a) *$(X, T)$ is distal.*

(b) *$(X^I, T)$ is pointwise almost periodic where $I$ is a set with at least two elements.*

## 3. Some results on homomorphisms of pointed minimal sets

Let $\beta T$ be the Stone-Cěch compactification of $T$. Then $(\beta T, e)$ is a universal point transitive flow. It is also clear that $\beta T$ is an enveloping semigroup for $X$, whenever $X$ is a flow with acting group $T$. Now let $M$ be a fixed minimal right ideal in $\beta T$. Then $(M, T)$ is a universal minimal set. We choose a distinguished idempotent $u$ in $J(M) = J$

and let $\mathcal{G}$ denote the group $Mu$. Given a minimal set $X$, we choose a distinguished onto homomorphism $\gamma : M \to X$ and let $\gamma(u) = x_0$. Then $x_0u = x_0$. Thus $x_0 \in Xu$.

Now we define the Ellis group $\mathcal{G}(X, x_0)$ and the quasi-Ellis group $\mathcal{S}(X, x_0)$ as follows ;

$$\mathcal{G}(X, x_0) = \{\alpha \in \mathcal{G} \mid x_0\alpha = x_0\} \text{ ([5])}$$

$$\mathcal{S}(X, x_0) = \{\alpha \in \mathcal{G} \mid h(x_0)\alpha = x_0 \text{ for some h} \in \text{A(X)}\}.$$

Clearly $\mathcal{G}(X, x_0) \subset \mathcal{S}(X, x_0)$, and $\mathcal{G}(X, x_0)$ and $\mathcal{S}(X, x_0)$ are subgroups of $\mathcal{G}$.

Let $G = A(M)$. Given a homomorphism $\gamma : M \to X$ we define the subgroups $G(X, \gamma)$ and $S(X, \gamma)$ of $G$ as follows(see [1], [8]);

$$G(X, \gamma) = \{\theta \in G \mid \gamma \circ \theta = \gamma\}$$

$$S(X, \gamma) = \{\theta \in G \mid h \circ \gamma \circ \theta = \gamma \text{ for some h} \in \text{A(X)}\}.$$

Lemma 3.1. *The following are true :*

(1) *Let $\alpha \in \mathcal{G}(X, x_0)$ and let $\theta : M \to M$ be the map with $\theta(u) = \alpha$. Then $\theta \in G(X, \gamma)$.*

(2) *Let $\theta \in G(X, \gamma)$ and let $\theta(u) = \alpha$. Then $\gamma(\alpha) = x_0$ and $\alpha \in \mathcal{G}(X, x_0)$.*

*Proof.* (1) Let $\alpha \in \mathcal{G}(X, x_0)$ and define the map $\theta : M \to M$ by $\theta(u) = \alpha$. Given elements $p$ and $q$ in $M$ with $up = uq$, we also have $\alpha p = \alpha q$. This imples that $\theta$ is a unique homomorphism by Lemma 2.3. Hence it follows from [3, Proposition 14] that $\theta \in A(M)$.

Moreover, $\gamma \circ \theta(u) = \gamma(\theta(u)) = \gamma(\alpha) = \gamma(u\alpha) = \gamma(u)\alpha = x_0\alpha = x_0 = \gamma(u)$. Thus $\theta \in G(X, \gamma)$.

(2) Let $\theta \in G(X, \gamma)$ and let $\theta(u) = \alpha$. Then $\gamma(\alpha) = \gamma(\theta(u)) = \gamma \circ \theta(u) = \gamma(u) = x_0$ and hence $x_0\alpha = \gamma(u)\alpha = \gamma(u\alpha) = \gamma(\alpha) = x_0$. But since $\alpha = \theta(u) = \theta(u)u = \alpha u \in \mathcal{G}$, it follows that $\theta \in A(M)$. $\square$

Theorem 3.2. *The groups $\mathcal{G}$ and $G$ are isomorphic.*

*Proof.* Define $\Phi : G \to \mathcal{G}$ by $\Phi(\theta) = \theta(u)$ for all $\theta \in G$ . Since $\theta(u) = \theta(u)u \in \mathcal{G}$, $\Phi$ is well defined. Let $\Phi(\theta_1) = \Phi(\theta_2)$. Since $\theta_1(u) = \theta_2(u)$, it follows from the minimality of $M$ that $\theta_1 = \theta_2$. This means that $\Phi$ is injective. Now let $\alpha \in \mathcal{G}$. Then there exists $p \in M$ with $\alpha = pu$ and we can choose $\theta \in G$ with $\theta(u) = p$ by Lemma 3.1 (1). Thus $\Phi(\theta) = \theta(u) = \theta(u)u = pu = \alpha$ whence $\Phi$ is surjective. Finally let $\theta, \eta \in G$. Then $\Phi(\theta\eta) = \theta\eta(u) = \theta(u\eta(u)) = \theta(u)\eta(u) = \Phi(\theta)\Phi(\eta)$, which means that $\Phi$ is a homomorphism. Therefore $\Phi$ is a isomorphism whence $\mathcal{G}$ and $G$ are isomorphic. $\square$

THEOREM 3.3. *The following are true :*
(1) $\mathcal{G}(X, x_0)$ *and* $G(X, \gamma)$ *are isomorphic.*
(2) $\mathcal{S}(X, x_0)$ *and* $S(X, \gamma)$ *are isomorphic.*

*Proof.* (1) By Lemma 3.1 (2), $\Phi|_{G(X,\gamma)} : G(X, \gamma) \to \mathcal{G}(X, x_0)$ is well defined. Also $\Phi|_{G(X,\gamma)}$ is surjective by Lemma 3.1 (1). Therefore it is trivial the fact that $\mathcal{G}(X, x_0)$ and $G(X, \gamma)$ are isomorphic.
(2) This follows from Definition and 3.3 (1). $\square$

The next theorems follow from Theorem 3.3.

THEOREM 3.4. ([8]) *The following are true :*
(1) $\mathcal{G}(X, x_0)$ *is a normal subgroup of* $\mathcal{S}(X, x_0)$.
(2) $G(X, \gamma)$ *is a normal subgroup of* $S(X, \gamma)$.

THEOREM 3.5. ([5]) *Let* $\pi : (X, x_0) \to (Y, y_0)$ *be a homomorphism of pointed minimal sets. Then the following are true :*
(1) $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$.
(2) $G(X, \gamma) \subset G(Y, \pi \circ \gamma)$.

REMARK 3.6. Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets and let $\gamma : M \to X$ be a fixed homomorphism. Then the following are true :
(1) The groups $\mathcal{G}$, $\mathcal{G}(X, x_0)$, and $\mathcal{S}(X, x_0)$ can be identified with the groups $G$, $G(X, \gamma)$, and $S(X, \gamma)$, respectively.
(2) The group $\mathcal{G}(Y, y_0)$ can be identified with the group $G(Y, \pi\circ\gamma)$.

THEOREM 3.7. *Let* $(X, T)$ *be a minimal set and let* $x_0 \in Xu$. *Then* $\beta \in \mathcal{G} - \mathcal{G}(X, x_0)$ *if and only if* $(x_0, x_0\beta) \notin P(X)$.

*Proof.* Let $\beta \in \mathcal{G} - \mathcal{G}(X, x_0)$ and suppose $(x_0, x_0\beta) \in P(X)$. Then there exists $q \in M$ with $x_0 q = x_0\beta q$. Since $qM = M$, it follows that there exists $r \in M$ with $qr = u$. Hence $x_0 = x_0 u = x_0 qr = x_0\beta qr = x_0\beta u = x_0\beta$. This is a contradiction because $\beta \notin \mathcal{G}(X, x_0)$.
Now let $\beta \in \mathcal{G}$ and suppose $(x_0, x_0\beta) \notin P(X)$. Then $x_0 p \neq x_0\beta p$ for all $p \in \beta T$. Hence $x_0 = x_0 u \neq x_0\beta u = x_0\beta$. This implies $\beta \notin \mathcal{G} - \mathcal{G}(X, x_0)$. $\square$

LEMMA 3.8. *Let* $(X, T)$ *be a flow. Then the following are true :*
(1) *If* $(x, x') \notin P(X)$, *then* $xu \neq x'u$.

(2) *Let* $(X, T)$ *be a minimal set and let* $x, x' \in Xu$. *Then there exists* $\beta \in \mathcal{G}$ *such that* $x = x'\beta$.

*Proof.* (1) Suppose $xu = x'u$. Then $xp = xup = x'up = x'p$ for all $p \in \mathcal{G}$ and hence $(x, x') \in P(X)$.

(2) Let $x, x' \in Xu$. Since $x'M = X$, it follows that there exists $q \in M$ such that $x'q = x$. Set $qu = \beta$. Then $\beta \in \mathcal{G}$ and $x'\beta = x'qu = xu = x$. $\square$

Remark 3.9. (1) Note that if $p \in M$, $p$ has a unique decomposition as $p = \alpha v$ where $\alpha \in \mathcal{G}$ and $v \in J$ by Lemma 2.1 (5).

(2) If $x, x' \in Xu$, then there exists $\beta \in \mathcal{G}$ such that $\mathcal{G}(X, x') = \beta\mathcal{G}(X, x)\beta^{-1}$. In fact, if $\beta \in \mathcal{G}$ with $x'\beta = x$ and $\alpha \in \mathcal{G}(X, x)$, then $\beta\alpha\beta^{-1} \in \mathcal{G}(X, x')$ by Lemma 3.8 (2). Also, if $\delta \in \mathcal{G}(X, x')$, then it is immediate that $\delta \in \beta\mathcal{G}(X, x)\beta^{-1}$.

Theorem 3.10. *Let* $\pi : (X, x_0) \to (Y, y_0)$ *be a homomorphism of pointed minimal sets. Then the following are true :*

*(1) If* $P(x_0) = X$, *then* $P(X) = X \times X$.

*(2) If* $P(x_0) = X$, *then* $P(y_0) = Y$.

*(3) If* $\mathcal{G}(X, x_0) = \mathcal{G}$, *then* $P(X) = X \times X$ *and* $P(Y) = Y \times Y$.

*Proof.* (1) Let $x, x' \in X$. Then we have from Lemma 2.2 and $P(x_0) = X$ that there exist $v, w \in J$ such that $x = x_0 v$ and $x' = x_0 w$. Then $xw = (x_0 v)w = x_0 w = x'$. Therefore $(x, x') \in P(X)$ and hence $P(X) = X \times X$.

(2) Let $y \in Y$. Since $\pi$ is surjective, it follows that there exists $x \in X$ with $\pi(x) = y$. Since $P(x_0) = X$, we have that there exists $v \in J$ such that $x = x_0 v$. Then $y = \pi(x) = \pi(x_0 v) = y_0 v$. Thus $y \in P(y_0)$. This means that $P(y_0) = Y$.

(3) Let $\mathcal{G}(X, x_0) = \mathcal{G}$, $x \in X$, and $y \in Y$. Since $X$ is minimal, it follows that there exists $p \in M$ with $x = x_0 p$. Also we have from Remark 3.9 (1) that there exist $\alpha \in \mathcal{G}$, $v \in J$ such that $p = \alpha v$. Since $\mathcal{G}(X, x_0) = \mathcal{G}$, it follows that $x = x_0 p = x_0 \alpha v = x_0 v$. Thus $P(x_0) = X$ and hence $P(y_0) = Y$ by (2). It follows from (1) that $P(X) = X \times X$ and $P(Y) = Y \times Y$. $\square$

Theorem 3.11. ([5]) *Let* $\pi : (X, x_0) \to (Y, y_0)$ *be a homomorphism of pointed minimal sets. Then the following are equivalent :*

(a) $\pi$ *is proximal.*

(b) $\mathcal{G}(X, x_0) = \mathcal{G}(Y, y_0)$

(c) *Given* $y \in Y$, $\pi^{-1}(y) \subset xJ(M)$ *for all* $x \in \pi^{-1}(y)$.

The following theorem will show that in the case of distality, the group determines the flow homomorphism. We prove J. Auslander's result as a corollary.

THEOREM 3.12. *Let $X, Y$ be minimal sets and let $x_0 \in Xu$, $y_0 \in Yu$, and $Y$ distal. If $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$, then there exists a homomorphism $\pi : (X, x_0) \to (Y, y_0)$.*

*Proof.* Let $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$, $(x_0, y_0) = z_0$, and $\overline{z_0 T} = Z$. Given $\alpha \in \mathcal{G}$ with $z_0\alpha = z_0$, we have that $(x_0, y_0)\alpha = (x_0, y_0)$ whence $\alpha \in \mathcal{G}(X, x_0)$ and $\alpha \in \mathcal{G}(Y, y_0)$. Thus $\alpha \in \mathcal{G}(X, x_0)$. Now let $\alpha \in \mathcal{G}(X, x_0)$. Since $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$, it follows from the Definition that $z_0\alpha = z_0$. This implies that $\mathcal{G}(X, x_0) = \mathcal{G}(Z, z_0)$. Therefore, by Theorem 3.11, there exists a proximal homomorphism $\psi : (Z, z_0) \to (X, x_0)$. Define $\phi : (Z, z_0) \to (Y, y_0)$ by $\phi(z_0) = y_0$. Then $\phi$ is a unique homomorphism by Lemma 2.3. Now define $\pi : (X, x_0) \to (Y, y_0)$ by $\pi(x_0) = \phi(z_0)$. If $\psi(z_1) = x_0$, then $(z_0, z_1) \in P(Z)$ whence $(\phi(z_0), \phi(z_1)) \in P(Y)$. Since $Y$ is distal, it follows Lemma 2.4 that $\phi(z_0) = \phi(z_1)$. Thus $\pi$ is a well defined homomorphism such that $\pi \circ \psi = \phi$. □

COROLLARY 3.13. ([2]) *Let $X, Y$ be minimal sets and let $x_0 \in Xu$, $y_0 \in Yu$, and $Y$ distal. Then $\mathcal{G}(X, x_0) \subset \mathcal{G}(Y, y_0)$ if and only if there exists a homomorphism $\pi : (X, x_0) \to (Y, y_0)$.*

*Proof.* This follows from Theorem 3.5 and Theorem 3.12. □

In [7], Song proved the following theorems :

THEOREM 3.14. *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets. Then $\pi$ is regular if and only if $\mathcal{G}(Y, y_0) \subset \mathcal{S}(X, x_0)$.*

THEOREM 3.15. *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets. Then the following are true :*

(1) *If $\pi$ is regular, then $\mathcal{G}(X, x_0)$ is a normal subgroup of $\mathcal{G}(Y, y_0)$.*

(2) *Let $\pi$ be regular. For each $y \in Y$ and $x \in \pi^{-1}(y)$, there exists $\phi \in A(X)$ such that $\phi(x) \in x'J$ for all $x' \in \pi^{-1}(y)$.*

J. Auslander and S. Glasner proved the following theorem :

THEOREM 3.16. ([2], [5]) *Let $\pi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal sets. Then the following are equivalent :*

(a) *$\pi$ is distal.*

(b) *If $y \in Y$ and $v \in J$ such that $yv = y$, then $\pi^{-1}(y)v = \pi^{-1}(y)$.*

(c) *If* $y \in Y$, *then* $\pi^{-1}(yp) = \pi^{-1}(y)p$ *for all* $p \in M$.

(d) *Given* $y \in Y$ *and* $p \in M$ *with* $y_0 p = y$, *we have that* $\pi^{-1}(y) = x_0 \mathcal{G}(Y, y_0)p$.

Remark 3.17. (1) If $X$ is proximal, then $\mathcal{G}(X, x_0) = \mathcal{G}(Y, y_0) = \mathcal{S}(X, x_0)$. Note that if $X$ is proximal and minimal, then the only homomorphism of $X$ into $X$ is the identity (see 1 in [4]).

(2) Note that a homomorphism is both proximal and distal if and only if it is an isomorphism.

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