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HIERARCHICAL ERROR ESTIMATORS FOR LOWEST-ORDER MIXED FINITE ELEMENT METHODS

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Abstract. In this work we study two a posteriori error estimators of hierarchical type for lowest-order mixed finite element methods. One estimator is computed by solving a global defect problem based on the splitting of the lowest-order Brezzi–Douglas–Marini space, and the other estimator is locally computable by applying the standard localization to the first estimator. We establish the reliability and efficiency of both estimators by comparing them with the standard residual estimator. In addition, it is shown that the error estimator based on the global defect problem is asymptotically exact under suitable conditions.

1. Introduction

In this paper we consider the second-order elliptic problem on a bounded polygonal domain

$$\begin{cases} -\operatorname{div}(a\nabla u) = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = -g & \text{on } \partial\Omega \end{cases} \tag{1}$$

for given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\partial\Omega)$. It is assumed that the coefficient a is a symmetric, bounded and uniformly positive definite matrix-valued function. The Dirichlet boundary condition is assumed only for the sake

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of simplicity and subsequent results are easily extended to more general boundary conditions.

When the primary interest is accurate approximation of the vector variable $\boldsymbol{\sigma} = -a\nabla u$, the following mixed formulation of (1) is often preferred:

find $(\boldsymbol{\sigma}, u) \in H(\mathrm{div}, \Omega) \times L^2(\Omega)$ such that

$$\begin{cases} (a^{-1}\boldsymbol{\sigma}, \boldsymbol{\tau})_\Omega - (\mathrm{div}\, \boldsymbol{\tau}, u)_\Omega = \langle g, \boldsymbol{\tau} \cdot \boldsymbol{n} \rangle_{\partial\Omega} & \forall \boldsymbol{\tau} \in H(\mathrm{div}, \Omega) \\ (\mathrm{div}\, \boldsymbol{\sigma}, v)_\Omega = (f, v)_\Omega & \forall v \in L^2(\Omega), \end{cases} \tag{2}$$

where $H(\mathrm{div}, \Omega)$ stands for the space of square-integrable vector-valued functions whose divergences are also square-integrable. We have particularly in mind two lowest-order mixed finite element methods(MFEMs) on triangular meshes for the mixed formulation (2): the standard MFEM using the lowest-order Raviart–Thomas element [3, 12] and the multipoint-flux MFEM using the lowest-order Brezzi–Douglas–Marini element [15].

Adaptive refinement based on a posteriori error estimators is now a well-established tool for efficient implementation of finite element methods. In the past two decades much effort has been devoted to development of a posteriori error estimators for mixed finite element methods of (2); see, for example, [1, 2, 4, 5, 6, 7, 8, 10, 11, 13, 14] and references therein. We also refer to [16] for comparison of four different kinds of error estimators. For the reliability and efficiency of an error estimator, it is required that the ratio of the estimated error to the actual error stays between two positive bounds independent of the mesh size (up to higher order terms). For some error estimators this ratio becomes unity as the mesh is refined under favorable circumstances. In such cases the error estimator is said to be asymptotically exact.

In this work we study two a posteriori error estimators of hierarchical type for the lowest-order mixed finite element methods mentioned above. The first estimator is based on solution of a global defect problem and the second estimator is obtained through standard localization of the first one. While the hierarchical error estimator of [16] targeted the vector error $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ in the $H(\mathrm{div})$-norm (and the scalar error $\|u - u_h\|_{0,\Omega}$), we aim at estimating the vector error $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ in the L^2-norm. It turns out that we can use a smaller surplus space for this aim and the resulting error estimators have lower computational costs. We will show that this simplification maintains the reliability and efficiency of the error estimators (under the saturation assumption for the multipoint-flux MFEM). Moreover, the first estimator based on the global defect problem is asymptotically exact under suitable conditions.

The rest of the paper is organized as follows. In the next section we introduce some notation and the mixed finite element methods. In Section 3 we present our hierarchical error estimators and then discuss their reliability and efficiency in Section 4. Finally, the asymptotic exactness of the error estimator based on the global defect problem is established in Section 5.

2. Preliminaries

Suppose that $\mathcal{T}_h$ is a shape-regular triangulation of $\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h} K$ into triangles with the mesh size $h = \max_{K\in\mathcal{T}_h} h_K$. For an element $K \in \mathcal{T}_h$, we denote the diameter of K by h_K and the set of three edges of K by $\mathcal{E}_K$. The collection of all edges of $\mathcal{T}_h$ is denoted by $\mathcal{E}_h = \{e\}$, and we set

$$\mathcal{E}_h^{\Omega} = \{e \in \mathcal{E}_h : e \subset \Omega\}, \qquad \mathcal{E}_h^{\partial} = \{e \in \mathcal{E}_h : e \subset \partial\Omega\}.$$

Throughout the paper, we denote by C (with or without a subscript) a generic positive constant independent of the mesh size h which may take different values in different places.

Let $P_k(K)$ denote the space of all polynomials on K of total degree $\leq k$ and let

$$W_k = \{v_h \in L^2(\Omega) : v_h|_K \in P_k(K) \ \ \forall K \in \mathcal{T}_h\}.$$

The Raviart–Thomas and Brezzi–Douglas–Marini spaces over $\mathcal{T}_h$ are defined by

$$\begin{aligned} RT_k &= \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) : \boldsymbol{\tau}|_K \in RT_k(K) \ \ \forall K \in \mathcal{T}_h\}, \\ BDM_k &= \{\boldsymbol{\tau} \in H(\mathrm{div};\Omega) : \boldsymbol{\tau}|_K \in (P_k(K))^2 \ \ \forall K \in \mathcal{T}_h\}, \end{aligned}$$

where

$$RT_k(K) := (P_k(K))^2 \oplus (x_1, x_2)P_k(K).$$

In this paper we will consider the following mixed finite element methods based on the mixed formulation (2):

(Raviart–Thomas MFEM): Find $(\boldsymbol{\sigma}_h, u_h) \in RT_0 \times W_0$ such that

$$\begin{cases} (a^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_\Omega - (\mathrm{div}\,\boldsymbol{\tau}_h, u_h)_\Omega = \langle g, \boldsymbol{\tau}_h \cdot \boldsymbol{n}\rangle_{\partial\Omega} & \forall \boldsymbol{\tau}_h \in RT_0 \\ (\mathrm{div}\,\boldsymbol{\sigma}_h, v_h)_\Omega = (f, v_h)_\Omega & \forall v_h \in W_0. \end{cases} \tag{3}$$

(Multipoint-flux MFEM): Find $(\boldsymbol{\sigma}_h, u_h) \in BDM_1 \times W_0$ such that

$$\tag{4} \begin{cases} (a^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_{h,\Omega} - (\operatorname{div} \boldsymbol{\tau}_h, u_h)_\Omega = \langle g, \boldsymbol{\tau}_h \cdot \boldsymbol{n} \rangle_{\partial\Omega} & \forall \boldsymbol{\tau}_h \in BDM_1 \\ (\operatorname{div} \boldsymbol{\sigma}_h, v_h)_\Omega = (f, v_h)_\Omega & \forall v_h \in W_0. \end{cases}$$

Here $(\cdot,\cdot)_\Omega$ (resp. $\langle \cdot,\cdot \rangle_{\partial\Omega}$) denotes the standard L^2 inner product over Ω (resp. $\partial\Omega$). We refer to [15] for more details on the multipoint-flux MFEM, where the bilinear form $(a^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_{h,\Omega}$ is defined by applying the 3-point trapezoidal rule to the local integral $(a^{-1}\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)_K$ and summing the results over all $K \in \mathcal{T}_h$. An important observation is that

$$\tag{5} (a^{-1}\boldsymbol{\xi}_h, \boldsymbol{\tau}_h)_{h,\Omega} = (a^{-1}\boldsymbol{\xi}_h, \boldsymbol{\tau}_h)_\Omega \qquad \forall \boldsymbol{\xi}_h \in RT_0$$

if a and $\boldsymbol{\tau}_h$ is piecewise constant over $\mathcal{T}_h$. Both of the mixed finite element methods defined above have the lowest order of convergence (cf. [3, 15])

$$\tag{6} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \le Ch \|\boldsymbol{\sigma}\|_{1,\Omega}$$

and satisfy the local conservation law

$$\operatorname{div} \boldsymbol{\sigma}_h = \overline{f},$$

where $\overline{f}$ represents the L^2 projection of f onto W_0.

Finally, we recall the following integration-by-parts formula for a domain $T \subset \mathbb{R}^2$

$$(\operatorname{rot} \boldsymbol{\tau}, v)_T - (\boldsymbol{\tau}, \mathbf{curl}\, v)_T = \langle \boldsymbol{\tau} \cdot \boldsymbol{t}_T, v \rangle_{\partial T}$$

which will be frequently used throughout the paper. Here $\boldsymbol{t}_T$ is the unit tangent vector on ∂T oriented in the counterclockwise orientation and the differential operators are defined as

$$\operatorname{rot} \boldsymbol{\tau} = \frac{\partial \tau_2}{\partial x_1} - \frac{\partial \tau_1}{\partial x_2}, \qquad \mathbf{curl}\, v = \left(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1} \right)$$

for a vector-valued function $\boldsymbol{\tau} = (\tau_1, \tau_2)$ and a scalar-valued function v.

3. Hierarchical Error Estimators

In this section we present two a posteriori error estimators of the hierarchical type for the vector error $\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}$ of the mixed finite element methods (3) and (4).

The first estimator is closely related to the global minimization problem

$$\min_{\varphi_h \in P_2^0} \|a^{-1/2}(\boldsymbol{\sigma}_h + \mathbf{curl}\,\varphi_h - \boldsymbol{\sigma})\|_{0,\Omega}, \tag{7}$$

where the trial function space is defined by

$$P_2^0 := \{\varphi \in H^1(\Omega) : \varphi|_K \in P_2(K) \text{ for all } K \in \mathcal{T}_h \text{ and } \varphi \text{ vanishes at vertices of } \mathcal{T}_h\}.$$

This problem was motivated in [9] by the desire to find a vector function $\boldsymbol{\sigma}_h + \mathbf{curl}\,\varphi_h \in BDM_1$ which is more accurate than $\boldsymbol{\sigma}_h$ and may be regarded as based on the hierarchical splitting

$$BDM_1 = RT_0 + \mathbf{curl}\,P_2^0.$$

By the standard argument the minimization problem (7) is reduced to the problem of finding $\psi_h \in P_2^0$ such that

$$(a^{-1}\,\mathbf{curl}\,\psi_h, \mathbf{curl}\,\varphi_h)_\Omega = -(a^{-1}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}), \mathbf{curl}\,\varphi_h)_\Omega \qquad \forall \varphi_h \in P_2^0.$$

By virtue of the equality $a^{-1}\boldsymbol{\sigma} = -\nabla u$ and the integration by parts, this equation becomes

$$\begin{aligned}(a^{-1}\,\mathbf{curl}\,\psi_h, \mathbf{curl}\,\varphi_h)_\Omega = &- (a^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\,\varphi_h)_\Omega \\ &+ \langle g, \mathbf{curl}\,\varphi_h \cdot \boldsymbol{n}\rangle_{\partial\Omega} \quad \forall \varphi_h \in P_2^0,\end{aligned} \tag{8}$$

which is exactly the *global defect problem* based on the above splitting of BDM_1. The matrix system arising from (8) is symmetric and positive definite, and moreover, is known to be well-conditioned with respect to the mesh size h. Hence it can be efficiently solved, e.g., by applying the conjugate gradient method (even without any preconditioning when a is smooth).

Now the error estimator is defined as

$$\eta_{HG} := \|a^{-1/2}\,\mathbf{curl}\,\psi_h\|_{0,\Omega},$$

where $\psi_h \in P_2^0$ is the solution of the global defect problem (8).

The locally computable error estimator is constructed by means of the well-known localization technique for the hierarchical surplus space P_2^0. Let $\theta_e \in P_2^0$ be the nodal basis function associated with the midpoint m_e of the edge $e \in \mathcal{E}_h$ (that is, $\theta_e(m_{e'}) = \delta_{e,e'}$) and let $\omega_e := \operatorname{supp}\theta_e$. Then we define the error estimator

$$\eta_{HL} := \left(\sum_{e\in\mathcal{E}_h} \|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e}^2\right)^{1/2},$$

where $\psi_e \in \operatorname{span}\{\theta_e\}$ is the solution of the *local defect problem*

(9)
$$(a^{-1}\,\mathbf{curl}\,\psi_e, \mathbf{curl}\,\varphi_e)_{\omega_e} = -(a^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\,\varphi_e)_{\omega_e} + \langle g, \mathbf{curl}\,\varphi_e \cdot \boldsymbol{n}\rangle_{\partial\Omega\cap\omega_e}$$

for all $\varphi_e \in \operatorname{span}\{\theta_e\}$.

REMARK 1. *Based on the hierarchical splitting of the first-order Raviart–Thomas element*

$$RT_1 = RT_0 + \mathbf{curl}\,P_2^0 + \widehat{RT}_1^1, \qquad W_1 = W_0 + \widehat{W}_1,$$

the hierarchical error estimator of [16] *(designed to estimate $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ in the $H(\mathrm{div})$-norm) has two contributions coming from $\mathbf{curl}\,P_2^0$ and $\widehat{RT}_1^1 \times \widehat{W}_1$. The global defect problem (8) was introduced in connection with $\mathbf{curl}\,P_2^0$, but instead of considering η_{HG}, it was replaced by the local defect problems (9) to obtain η_{HL} as the contribution coming from $\mathbf{curl}\,P_2^0$. The contribution from $\widehat{RT}_1^1 \times \widehat{W}_1$ is locally computable by solving 4×4 saddle-point problems. Because η_{HG} and η_{HL} are equivalent to the residual estimator of* [2] *(up to higher order terms) as shown below, we may exclude the contribution from $\widehat{RT}_1^1 \times \widehat{W}_1$ when estimating the L^2-norm of $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$, which implies that our error estimators have lower computational costs.*

The following equivalence of η_{HG} and η_{HL} can be established in a standard manner and is valid for both the Raviart–Thomas MFEM (3) and the multipoint-flux MFEM (4).

THEOREM 1. *There exist positive constants C_1 and C_2 such that*

$$C_1\eta_{HL} \le \eta_{HG} \le C_2\eta_{HL}.$$

Proof. Taking $\varphi_e = \psi_e$ in (9) and $\varphi_h = \psi_e$ in (8), we find that

$$\begin{aligned}\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e}^2 &= (a^{-1}\,\mathbf{curl}\,\psi_h, \mathbf{curl}\,\psi_e)_{\omega_e}\\ &\le \|a^{-1/2}\,\mathbf{curl}\,\psi_h\|_{0,\omega_e}\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e},\end{aligned}$$

from which the left inequality immediately follows.

To derive the right inequality, let $\psi_h = \sum_{e\in\mathcal{E}_h} \alpha_e\theta_e$ for $\alpha_e \in \mathbb{R}$. Taking $\varphi_h = \psi_h$ in (8) and using (9), we obtain

$$\begin{aligned}\|a^{-1/2}\,\mathbf{curl}\,\psi_h\|_{0,\Omega}^2 &= \sum_{e\in\mathcal{E}_h} \alpha_e\big\{-(a^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\,\theta_e)_{\omega_e} + \langle g, \mathbf{curl}\,\theta_e\cdot\boldsymbol{n}\rangle_{\partial\Omega\cap\omega_e}\big\}\\ &= \sum_{e\in\mathcal{E}_h} \alpha_e(a^{-1}\,\mathbf{curl}\,\psi_e, \mathbf{curl}\,\theta_e)_{\omega_e}\\ &\le \sum_{e\in\mathcal{E}_h} \alpha_e\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e}\|a^{-1/2}\,\mathbf{curl}\,\theta_e\|_{0,\omega_e}\\ &\le \eta_{HL}\bigg(\sum_{e\in\mathcal{E}_h} \alpha_e^2\|a^{-1/2}\,\mathbf{curl}\,\theta_e\|_{0,\omega_e}^2\bigg)^{1/2}.\end{aligned}$$

Now the right inequality is proved by invoking the well-known equivalence

$$\begin{aligned}C_3\sum_{e\in\mathcal{E}_h}\alpha_e^2\|a^{-1/2}\,\mathbf{curl}\,\theta_e\|_{0,\omega_e}^2 &\le \|a^{-1/2}\,\mathbf{curl}\,\psi_h\|_{0,\Omega}^2\\ &\le C_4\sum_{e\in\mathcal{E}_h}\alpha_e^2\|a^{-1/2}\,\mathbf{curl}\,\theta_e\|_{0,\omega_e}^2\end{aligned}$$

with some positive constants C_3 and C_4. This completes the proof. □

4. Reliability and Efficiency

4.1. Lower bounds. A global lower bound for η_{HG} can be directly derived from (8):

$$\begin{aligned}\|a^{-1/2}\,\mathbf{curl}\,\psi_h\|_{0,\Omega}^2 &= -(a^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\,\psi_h)_\Omega - \langle u, \mathbf{curl}\,\psi_h\cdot\boldsymbol{n}\rangle_{\partial\Omega}\\ &= (a^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), \mathbf{curl}\,\psi_h)_\Omega\\ &\le \|a^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega}\|a^{-1/2}\,\mathbf{curl}\,\psi_h\|_{0,\Omega}\end{aligned}$$

Likewise a local lower bound for η_{HL} can be derived from (9):

$$\begin{aligned}\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e}^2 &= -(a^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\,\psi_e)_{\omega_e} - \langle u, \mathbf{curl}\,\psi_e\cdot\boldsymbol{n}\rangle_{\partial\Omega\cap\omega_e}\\ &= (a^{-1}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h), \mathbf{curl}\,\psi_e)_{\omega_e}\\ &\le \|a^{-1/2}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\omega_e}\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e}\end{aligned}$$

Note that η_{HG} and η_{HL} provide guaranteed lower bounds for the global error $\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}$ and the local error $\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\omega_e}$, respectively. We remark that guaranteed upper bounds for the global error $\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}$ were studied in [1, 8, 13].

4.2. Upper bound for Raviart–Thomas MFEM. To derive the upper bounds for η_{HG} and η_{HL}, we will compare them with the residual estimator of Alonso [2]. From now on it is assumed that the coefficient a is piecewise constant over $\mathcal{T}_h$. Then we have $\operatorname{rot}(a^{-1}\boldsymbol{\sigma}_h)|_K = 0$ for every $K \in \mathcal{T}_h$ and the residual estimator for the Raviart–Thomas MFEM (3) is given by

$$\eta_R := \Bigg(\sum_{e\in\mathcal{E}_h^\Omega} h_e \|[\![a^{-1}\boldsymbol{\sigma}_h \cdot \boldsymbol{t}]\!]\|_{0,e}^2 + \sum_{e\in\mathcal{E}_h^\partial} h_e \|a^{-1}\boldsymbol{\sigma}_h \cdot \boldsymbol{t} - \tfrac{dg}{ds}\|_{0,e}^2 + \sum_{T\in\mathcal{T}_h} h_T^2 \|f - \overline{f}\|_{0,T}^2 \Bigg)^{1/2},$$

where h_e is the length of the edge e, $\boldsymbol{t}|_e$ is a fixed unit tangent vector on e, $[\![w]\!]|_e$ is the jump of w across $e \in \mathcal{E}_h^\Omega$, and $\frac{dw}{ds}\big|_e$ is the derivative of $w|_e$ in the direction of $\boldsymbol{t}|_e$ on e. The following local equivalence of η_{HL} and η_R was established in [16] for the Poisson equation with the homogeneous Dirichlet boundary data. It is straightforward to extend it to the piecewise constant tensor coefficient a and the nonhomogeneous Dirichlet boundary data g.

Theorem 2. *Let ψ_e be the solution of (9). Then we have for every $e \in \mathcal{E}_h^\Omega$*

$$\begin{aligned} C_1 \|a^{-1/2} \operatorname{\mathbf{curl}} \psi_e\|_{0,\omega_e}^2 &\le h_e \|[\![a^{-1}\boldsymbol{\sigma}_h \cdot \boldsymbol{t}]\!]\|_{0,e}^2 \\ &\le C_2\big(\|a^{-1/2} \operatorname{\mathbf{curl}} \psi_e\|_{0,\omega_e}^2 + h_e^3 \|[\![\overline{f}(\boldsymbol{t}^T a^{-1}\boldsymbol{t})]\!]\|_{0,e}^2\big) \end{aligned}$$

and for every $e \in \mathcal{E}_h^\partial$

$$\begin{aligned} C_1 \|a^{-1/2} \operatorname{\mathbf{curl}} \psi_e\|_{0,\omega_e}^2 &\le h_e \|a^{-1}\boldsymbol{\sigma}_h \cdot \boldsymbol{t} - \tfrac{dg}{ds}\|_{0,e}^2 \\ &\le C_2\big(\|a^{-1/2} \operatorname{\mathbf{curl}} \psi_e\|_{0,\omega_e}^2 + h_e^3 \|\overline{f}(\boldsymbol{t}^T a^{-1}\boldsymbol{t})\|_{0,e}^2 \\ &\qquad + h_e^3 \|\tfrac{d^2 g}{ds^2}\|_{0,e}^2\big). \end{aligned}$$

Proof. Integration by parts in (9) gives for $e \in \mathcal{E}_h^\Omega$

$$(a^{-1} \operatorname{\mathbf{curl}} \psi_e, \operatorname{\mathbf{curl}} \varphi_e)_{\omega_e} = \langle [\![a^{-1}\boldsymbol{\sigma}_h \cdot \boldsymbol{t}]\!], \varphi_e \rangle_e. \tag{10}$$

Taking $\varphi_e = \psi_e$ and using the inequality $\|\psi_e\|_{0,e} \le C h_e^{1/2} \|\nabla \psi_e\|_{0,\omega_e}$, we obtain

$$\begin{aligned}\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|^2_{0,\omega_e} &= \langle [\![a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t}]\!], \psi_e\rangle_e \\ &\le \|[\![a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t}]\!]\|_{0,e}\cdot C h_e^{1/2}\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e},\end{aligned}$$

which proves the left inequality for $e \in \mathcal{E}_h^{\Omega}$. Similarly, we obtain for $e \in \mathcal{E}_h^{\partial}$

$$(a^{-1}\,\mathbf{curl}\,\psi_e, \mathbf{curl}\,\varphi_e)_{\omega_e} = \langle a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t} - \tfrac{dg}{ds}, \varphi_e\rangle_e, \tag{11}$$

and the same proof leads to the left inequality for $e \in \mathcal{E}_h^{\partial}$.

On the other hand, following the proof of [16, Theorem 7.1], we can derive for $e \in \mathcal{E}_h^{\Omega}$

$$h_e\|[\![a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t}]\!]\|^2_{0,e} \le C h_e^2\big|[\![a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t}]\!]|_e(m_e)\big|^2 + C h_e^3\|[\![\,\overline{f}(\boldsymbol{t}^T a^{-1}\boldsymbol{t})]\!]\|^2_{0,e}$$

and for $e \in \mathcal{E}_h^{\partial}$

$$\begin{aligned}h_e\|a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t} - \tfrac{dg}{ds}\|^2_{0,e} &\le C h_e^2\big|a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t}(m_e) - \tfrac{1}{h_e}\textstyle\int_e \tfrac{dg}{ds}\,ds\big|^2 \\ &\quad + C h_e^3\|\,\overline{f}(\boldsymbol{t}^T a^{-1}\boldsymbol{t})\|^2_{0,e} + C h_e\|\tfrac{dg}{ds} - \tfrac{1}{h_e}\textstyle\int_e \tfrac{dg}{ds}\,ds\|^2_{0,e}.\end{aligned}$$

Furthermore, Simpson's rule and (10) yields for $e \in \mathcal{E}_h^{\Omega}$

$$\begin{aligned}h_e\big|[\![a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t}]\!]|_e(m_e)\big| &= \frac{3}{2}\big|\langle [\![a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t}]\!], \theta_e\rangle_e\big| \\ &\le \frac{3}{2}\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e}\|a^{-1/2}\,\mathbf{curl}\,\theta_e\|_{0,\omega_e} \\ &\le C\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e},\end{aligned}$$

and (11) yields for $e \in \mathcal{E}_h^{\partial}$

$$\begin{aligned}h_e\big|a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t}(m_e) - \tfrac{1}{h_e}\textstyle\int_e \tfrac{dg}{ds}\,ds\big| &= \frac{3}{2}\big|\langle a^{-1}\boldsymbol{\sigma}_h\cdot\boldsymbol{t} - \tfrac{dg}{ds} + \tfrac{dg}{ds} - \tfrac{1}{h_e}\textstyle\int_e \tfrac{dg}{ds}\,ds, \theta_e\rangle_e\big| \\ &\le C\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e} + C h_e^{1/2}\|\tfrac{dg}{ds} - \tfrac{1}{h_e}\textstyle\int_e \tfrac{dg}{ds}\,ds\|_{0,e} \\ &\le C\|a^{-1/2}\,\mathbf{curl}\,\psi_e\|_{0,\omega_e} + C h_e^{3/2}\|\tfrac{d^2 g}{ds^2}\|_{0,e},\end{aligned}$$

where we used the estimates $\|\nabla\theta_e\|_{0,\omega_e} \le C$, $\|\theta_e\|_{0,e} \le C h_e^{1/2}$ and then the Poincaré inequality $\|w - \frac{1}{h_e}\int_e w\,ds\|_{0,e} \le C h_e\|\frac{dw}{ds}\|_{0,e}$. Combining the results above yields the right inequalities. □

The extra terms on the right inequalities of Theorem 2 as well as $\big(\sum_{T\in\mathcal{T}_h} h_T^2\|f - \overline{f}\|^2_{0,T}\big)^{1/2}$ are of higher order if f and g are piecewise smooth and thus become negligible for sufficiently small h. As a corollary of Theorems 1–2, we also obtain the global equivalence of η_{HG} and η_R up to higher order terms. Since η_R is reliable (and efficient) for estimating $\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}$, this implies that η_{HG} and η_{HL} are reliable (and efficient) for the same error up to higher order terms.

4.3. Upper bound for Multipoint-Flux MFEM. The argument of the preceding subsection does not seem to apply to the multipoint-flux MFEM (4). We are thus led to make the following saturation assumption: there exists a positive constant $\gamma < 1$ independent of the mesh size h such that

$$\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_2)\|_{0,\Omega} \leq \gamma \|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}, \tag{12}$$

where $\boldsymbol{\sigma}_2$ is the vector solution of the lowest-order Brezzi–Douglas–Marini MFEM given by

(BDM-MFEM): Find $(\boldsymbol{\sigma}_2, u_2) \in BDM_1 \times W_0$ such that

$$\begin{cases} (a^{-1}\boldsymbol{\sigma}_2, \boldsymbol{\tau}_h)_\Omega - (\operatorname{div} \boldsymbol{\tau}_h, u_2)_\Omega = \langle g, \boldsymbol{\tau}_h \cdot \boldsymbol{n} \rangle_{\partial\Omega} & \forall \boldsymbol{\tau}_h \in BDM_1 \\ (\operatorname{div} \boldsymbol{\sigma}_2, v_h)_\Omega = (f, v_h)_\Omega & \forall v_h \in W_0. \end{cases} \tag{13}$$

This assumption seems to be reasonable in view of the a priori error estimates (6) and

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_2\|_{0,\Omega} \leq Ch^2 \|\boldsymbol{\sigma}\|_{2,\Omega},$$

and was also made in [2] for the error estimator of the Bank–Weiser type.

THEOREM 3. *Under the saturation assumption (12), there exists a constant $C > 0$ such that*

$$\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \leq C\eta_{HG}.$$

The same result holds as well for η_{HL}.

Proof. Thanks to Theorem 1, it suffices to prove the result for η_{HG}. Combining (12) with the triangle inequality

$$\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \leq \|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_2)\|_{0,\Omega} + \|a^{-1/2}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_h)\|_{0,\Omega},$$

we obtain

$$\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \leq \frac{1}{1-\gamma} \|a^{-1/2}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_h)\|_{0,\Omega}. \tag{14}$$

Since $\operatorname{div} \boldsymbol{\sigma}_2 = \operatorname{div} \boldsymbol{\sigma}_h = \overline{f}$, there exists a continuous piecewise quadratic function φ_h such that

$$\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_h = \mathbf{curl}\, \varphi_h.$$

Let $I_1\varphi_h$ be the continuous piecewise linear nodal interpolant of φ_h satisfying

$$\|\nabla(I_1\varphi_h)\|_{0,\Omega} \le C\|\nabla\varphi_h\|_{0,\Omega}. \tag{15}$$

Then we have $\mathbf{curl}(I_1\varphi_h) \in RT_0$ and $\varphi_h - I_1\varphi_h \in P_2^0$, and it follows by the first equations of (13) and (4), (5), (8) and (15) that

$$\begin{aligned}\|a^{-1/2}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_h)\|_{0,\Omega}^2 &= (a^{-1}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_h), \mathbf{curl}\,\varphi_h)_\Omega \\ &= \langle g, \mathbf{curl}\,\varphi_h \cdot \boldsymbol{n}\rangle_{\partial\Omega} - (a^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}\,\varphi_h)_\Omega \\ &= \langle g, \mathbf{curl}(\varphi_h - I_1\varphi_h)\cdot \boldsymbol{n}\rangle_{\partial\Omega} - (a^{-1}\boldsymbol{\sigma}_h, \mathbf{curl}(\varphi_h - I_1\varphi_h))_\Omega \\ &= (a^{-1}\,\mathbf{curl}\,\psi_h, \mathbf{curl}(\varphi_h - I_1\varphi_h))_\Omega \\ &\le C\|a^{-1/2}\,\mathbf{curl}\,\psi_h\|_{0,\Omega}\|\nabla(\varphi_h - I_1\varphi_h)\|_{0,\Omega} \\ &\le C\|a^{-1/2}\,\mathbf{curl}\,\psi_h\|_{0,\Omega}\|\,\mathbf{curl}\,\varphi_h\|_{0,\Omega} \\ &\le C\|a^{-1/2}\,\mathbf{curl}\,\psi_h\|_{0,\Omega}\|a^{-1/2}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_h)\|_{0,\Omega},\end{aligned}$$

which gives

$$\|a^{-1/2}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_h)\|_{0,\Omega} \le C\eta_{HG}.$$

Now the proof is completed by combining the last result and (14). □

REMARK 2. *The above proof may be used to derive the same upper bound for the Raviart–Thomas MFEM (3) under the saturation assumption (12). In particular, this removes the extra higher order terms of Theorem 2 derived without the saturation assumption (12).*

5. Asymptotic Exactness

In this section we show that the error estimator η_{HG} based on the global defect problem (8) is asymptotically exact under suitable conditions. For this purpose we need the Fortin projection $\Pi_h : (H^1(\Omega))^2 \to RT_0$ defined by

$$\int_e \Pi_h\boldsymbol{\tau}\cdot\boldsymbol{n}_e\,ds = \int_e \boldsymbol{\tau}\cdot\boldsymbol{n}_e\,ds \qquad \forall e \in \mathcal{E}_h,$$

where $\boldsymbol{n}_e$ denotes a unit normal vector to e.

THEOREM 4. *Assume that* $\boldsymbol{\sigma} \in (H^{1+\rho}(\Omega))^2$ *and* $\|a^{-1/2}(\boldsymbol{\sigma}_h - \Pi_h\boldsymbol{\sigma})\|_{0,\Omega} = O(h^{1+\rho})$ *for some* $\rho > 0$. *Then we have*

$$\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} = \eta_{HG} + O(h^{1+\rho}).$$

In addition, if the non-degeneracy condition $\|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \geq Ch$ *holds for some constant* $C > 0$*, then*

$$\left| \frac{\eta_{HG}}{\|a^{-1/2}((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}} - 1 \right| = O(h^\rho).$$

Proof. Let $\psi_h \in P_2^0$ be the solution of (8). Then the proof of [9, Theorem 4.1] yields

$$\|a^{-1/2}(\boldsymbol{\sigma}_h + \mathbf{curl}\,\psi_h - \boldsymbol{\sigma})\|_{0,\Omega} = O(h^{1+\rho}),$$

and consequently,

$$\begin{aligned} \big| \|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} - \eta_{HG} \big| &= \big| \|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} - \|a^{-1/2}\,\mathbf{curl}\,\psi_h\|_{0,\Omega} \big| \\ &\leq \|a^{-1/2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \mathbf{curl}\,\psi_h)\|_{0,\Omega} \\ &\leq Ch^{1+\rho}, \end{aligned}$$

which gives the first result. The second result is a direct consequence of the first result. □

REMARK 3. *It was proved in* [5] *that* $\|\boldsymbol{\sigma}_h - \Pi_h \boldsymbol{\sigma}\|_{0,\Omega} = O(h^{1+\rho})$ *holds for the Raviart–Thomas MFEM (3) with* $\rho = \frac{1}{2}$ *under the restrictive conditions that the triangulation* $\mathcal{T}_h$ *is uniform and* $\boldsymbol{\sigma} \in (H^2(\Omega))^2$*. Numerical experiments suggest that this condition may be relaxed with a smaller value of* $\rho > 0$*. For the multi-point MFEM, this super-closeness has not been rigorously derived but may be conjectured by numerical results in* [15].

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