

ON THE ANTICYCLOTOMIC \mathbb{Z}_p -EXTENSION OF AN IMAGINARY QUADRATIC FIELD

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ABSTRACT. We prove that if a subfield of the Hilbert class field of an imaginary quadratic field k meets the anticyclotomic \mathbb{Z}_p -extension k_∞^a of k , then it is contained in k_∞^a . And we give an example of an imaginary quadratic field k with $\lambda_3(k_\infty^a) \geq 8$.

1. Introduction

An abelian extension L of k is called an anti-cyclotomic extension of k if it is Galois over \mathbb{Q} , and $Gal(k/\mathbb{Q})$ acts on $Gal(L/k)$ by -1 . For each prime number p , the compositum K of all \mathbb{Z}_p -extensions over k becomes a \mathbb{Z}_p^2 -extension, and K is the compositum of the cyclotomic \mathbb{Z}_p -extension k_∞^c and the anti-cyclotomic \mathbb{Z}_p -extension k_∞^a of k .

The layers k_n^c of the cyclotomic \mathbb{Z}_p -extension are well understood. Since the Hilbert class field of k is an anti-cyclotomic extension of k , determination of the first layer of the anti-cyclotomic \mathbb{Z}_p -extension becomes complicated as the p -rank of the p -Hilbert class field of k becomes larger. In the papers [3,5,6], using Kummer theory and class field theory, we constructed the first layer k_1^a of the anti-cyclotomic \mathbb{Z}_3 -extension of k under the assumption that the 3-part of Hilbert class field H_k of k is 3-elementary. A criterion on linearly disjointness of k_1^a and H_k over k is

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proved in [4] under the assumption. In this paper, we prove the criterion without the assumption. See Corollary 1 of this paper.

Contrary to the case of the cyclotomic \mathbb{Z}_p -extension, the lambda invariant $\lambda_p(k_\infty^a)$ of the anticyclotomic \mathbb{Z}_p -extension of an imaginary quadratic field is not well known. Few examples of computation of $\lambda_p(k_\infty^a)$ are given. Following the idea of Fujii [1], we give an example of k with $\lambda_3(k_\infty^a) \geq 8$.

2. Proof of Theorems

Let p be an odd prime number. Throughout this section, we denote by H_k, h_k, A_k , and M_k the p -part of Hilbert class field, the p -class number, p -part of ideal class group, and the maximal abelian p -extension of an imaginary quadratic field k unramified outside above p , respectively. The first layer of the anti-cyclotomic \mathbb{Z}_p -extension of k may be or may not be contained in the p -Hilbert class field of k . The following theorem and the criterion in [4] gives an answer for this question. We define $\text{rank}_{\mathbb{Z}/p\mathbb{Z}} A$ to be the dimension of A/A^p over $\mathbb{Z}/p\mathbb{Z}$ for any abelian group A . Note that $K \cap H_k = k_\infty^a \cap H_k$.

THEOREM 1. *Let $d \not\equiv 3 \pmod{9}$ be a square free positive integer, $k = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field. Let L be a subfield of H_k which satisfies the following properties:*

$$H_k \cap k_\infty^a = k_n^a \leq L (n \geq 1), \quad \text{Gal}(L/k) \text{ is cyclic.}$$

Then

$$L = k_n^a.$$

Proof. Assume that $k_n^a \neq L$. Then there exists a ramified extension of k of degree p which becomes unramified over k_∞^a . By class field theory, we see that

$$\text{Gal}(M_k/H_k) \simeq \left(\prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}} \right),$$

where $U_{1,\mathfrak{p}}$ is the local units of k which is congruent to 1 mod \mathfrak{p} . However, by the condition of Theorem 1, there is no p -torsion point in $\prod_{\mathfrak{p}|p} U_{1,\mathfrak{p}}$, which contradicts to the fact that the ramified extension of k of degree p exists. This completes the proof. \square

By Theorem 1 one can easily prove the following corollary, which was proved in [4] with the assumption that $A_{\mathbb{Q}(\sqrt{-d})}$ is 3-elementary, without the assumption. In fact, the following equivalence

$$H_k \cap k_\infty^a = k \iff \text{rank}_{\mathbb{Z}/p} X_{k,\chi} = 1 + \text{rank}_{\mathbb{Z}/p} A_k$$

in [4] holds without the assumption by Theorem 1. Here

$$X_k := \text{Gal}(M_k/k)/p\text{Gal}(M_k/k)$$

and $X_{k,\chi}$ be the χ -component of X_k for the nontrivial character χ of $\text{Gal}(k/\mathbb{Q})$.

COROLLARY 1. *Let $d \not\equiv 3 \pmod{9}$ be a square free positive integer, $k = \mathbb{Q}(\sqrt{-d})$ an imaginary quadratic field and k_∞^a the anti-cyclotomic \mathbb{Z}_3 -extension over k . Then*

$$H_k \cap k_\infty^a = k \iff \text{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{3d})} = \text{rank}_{\mathbb{Z}/3} A_{\mathbb{Q}(\sqrt{-d})}.$$

By following the idea of Fujii [1], we give an example of an imaginary quadratic field with large invariant $\lambda_3(k_\infty^a)$.

THEOREM 2.

$$\lambda_3(k_\infty^a) \geq 8,$$

where $k = \mathbb{Q}(\sqrt{-1423})$,

Proof. Denote by K_2^a the compositum of all \mathbb{Z}_3 -extensions of k_2^a . First note that the class number of $\mathbb{Q}(\sqrt{3 * 1423})$ is one. Hence, by Theorem 3 below, $H_k \subset k_\infty^a$. Since the class number of k is 9, $H_k = k_2^a$. By simple computation, we see that 3 stays prime in k . The definition of anti-cyclotomic extension and class field theory shows that \mathfrak{p}_3 , the prime of k above 3, splits completely in k_2^a . Note that the \mathbb{Z}_3 -rank of $\text{Gal}(K_2^a/k_2^a)$ is 10. Since the inertia group of primes of k_2^a above 3 is isomorphic to \mathbb{Z}_3^2 and K/k is abelian, the extension K_2^a/K is unramified everywhere. Hence the maximal abelian 3-extension of k_∞^a contains K_2^a , and the galois group of K_2^a over K is isomorphic to \mathbb{Z}_3^8 . This completes the proof. \square

The following theorem is given in [2].

THEOREM 3. *If $p = 3$ and $d \not\equiv 3 \pmod{9}$, then $H_k \subset k_\infty^a$ if and only if the class number of $\mathbb{Q}(\sqrt{3d})$ is not divisible by 3.*

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