Korean J. Math. **20** (2012), No. 4, pp. 433–439
http://dx.doi.org/10.11568/kjm.2012.20.4.433

# HARMONIC MAPPINGS WITH ANALYTIC FUNCTIONS

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Abstract. In this paper, we study harmonic, orientation-preserving, univalent mappings defined on $\Delta = \{z : |z| > 1\}$ that have real coefficients or starlike analytic functions and obtain some coefficients bounds.

## 1. Introduction

A continuous function $f = u + iv$ defined in a domain $D \subseteqq \mathbb{C}$ is harmonic in D if $u$ and $v$ are real harmonic in D. We consider complex-valued, harmonic, orientation-preserving, univalent mappings $f$ defined on $\Delta = \{z : |z| > 1\}$, that are normalized at infinity by $f(\infty) = \infty$. Such functions admit the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z|,$$

where

$$h(z) = \alpha z + \sum_{k=0}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \beta z + \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in $\Delta$ and $0 \leq |\beta| < |\alpha|$. In addition, $a = \overline{f_{\overline{z}}}/f_z$ is analytic

Received September 24, 2012. Revised October 30, 2012. Accepted November 5, 2012.
2010 Mathematics Subject Classification: 30C45, 30C50.
Key words and phrases: harmonic mapping.
This work was supported by a research grant from Seoul Women's University(2012).

and satisfies $|a(z)| < 1$. Also one can easily show that $|A|/2 \leq |\alpha|+|\beta|$ by using the bound $|s_1| \leq 1-|s_0|^2$ for analytic function $a = s_0+s_1z^{-1}+\cdots$ in $\Delta$ that are bounded by one. By applying an affine post-mapping to $f$ we may normalize $f$ so that $\alpha = 1, \beta = 0$, and $a_0 = 0$. Therefore let $\Sigma$ be the set of all harmonic, orientation-preserving, univalent mappings

$$f(z) = h(z) + \overline{g(z)} + A\log|z| \tag{1.1}$$

of $\Delta$, where

$$h(z) = z + \sum_{k=1}^{\infty} a_k z^{-k} \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^{-k}$$

are analytic in $\Delta$ and $A \in \mathbb{C}$. Hengartner and Schober[2] used the representation (1.1) to obtain coefficient bounds and distortion theorems. Some coefficients bounds for $f \in \Sigma$ also obtained by Jun[3].

In this article, we continue to investigate harmonic, orientation-preserving, univalent mappings $f$ in $\Sigma$ to get coefficients bounds for $f$ with some restrictions. In next section we consider univalent harmonic mappings $f \in \Sigma$ with real $A$ which have real coefficients and obtain estimates

$$|b_n - a_n| \leq n|1 + b_1 - a_1| \quad \text{for} \quad n \geq 2.$$

Also $f \in \Sigma$ with starlike analytic functions $h + g$ will be considered in section 3.

## 2. Harmonic mappings with real coefficients

THEOREM 2.1. *If $f \in \Sigma$ with real $A$ has real coefficients, then*

$$|b_n - a_n| \leq n|1 + b_1 - a_1| \quad \textit{for} \quad n \geq 2.$$

*Proof.* For $z = re^{i\theta}$, $r > 1$,

$$Im\{f(re^{i\theta})\} = \sum_{k=1}^{\infty} c_k \sin k\theta \tag{2.1}$$

where $c_1 = r + (b_1 - a_1)r^{-1}$ and $c_k = (b_k - a_k)r^{-k}$ for $k \geq 2$. Multiply $\sin n\theta$ to (2.1) and integrate from 0 to $\pi$, then we have

$$\begin{aligned}&\frac{2}{\pi}\int_0^{\pi} Im\{f(re^{i\theta})\}\sin n\theta\ d\theta \\ &= \frac{2}{\pi}\int_0^{\pi}\left(\sum_{k=1}^{\infty} c_k \sin k\theta\right)\sin n\theta\ d\theta = \frac{2}{\pi}\int_0^{\pi} c_n \sin^2 n\theta\ d\theta \\ &= c_n.\end{aligned}$$

From the relationship

$$|\sin(n+1)\theta| = |\sin n\theta\cos\theta + \cos n\theta\sin\theta| \leq |\sin n\theta| + |\sin\theta|,$$

we can show that $|\sin n\theta| \leq n|\sin\theta|$ by the mathematical induction. Thus we have

$$\begin{aligned}|c_n| &= \left|\frac{2}{\pi}\int_0^{\pi} Im\{f(re^{i\theta})\}\sin n\theta\ d\theta\right| \\ &\leq \frac{2}{\pi}\int_0^{\pi} |Im\{f(re^{i\theta})\}||\sin n\theta|\ d\theta \\ &\leq \frac{2n}{\pi}\int_0^{\pi} |Im\{f(re^{i\theta})\}|\sin\theta\ d\theta.\end{aligned} \tag{2.2}$$

The univalence of $f$ implies that $0 \neq f(re^{i\theta}) - f(re^{-i\theta})$ since $re^{i\theta} \neq re^{-i\theta}$ for $0 < \theta < \pi$. From $0 \neq f(re^{i\theta}) - f(re^{-i\theta}) = 2iIm\{f(re^{i\theta})\}$, we have $Im\{f(re^{i\theta})\} \neq 0$. Since $Im\{f(re^{i\theta})\}$ is a continuous function of $\theta$, it must be of same sign in the interval $0 < \theta < \pi$. Thus

$$\begin{aligned}&\frac{2}{\pi}\int_0^{\pi} |Im\{f(re^{i\theta})\}|\sin\theta\ d\theta \\ &= \left|\frac{2}{\pi}\int_0^{\pi} Im\{f(re^{i\theta})\}\sin\theta\ d\theta\right| \\ &= |c_1| \\ &= \left|r + \frac{b_1 - a_1}{r}\right|.\end{aligned}$$

Substituting this into (2.2), we have

$$|c_n| \leq n\left|r + \frac{b_1 - a_1}{r}\right|$$

where $c_1 = r + \frac{b_1 - a_1}{r}$ and $c_n = \frac{b_n - a_n}{r^n}$ for $n \geq 2$. Letting $r \to 1$, we obtain $|b_n - a_n| \leq n|1 + b_1 - a_1|$ for $n \geq 2$. $\square$

## 3. Starlike analytic functions

DEFINITION 3.1. *A function $H(z)$ is starlike if each radial line from the origin hits the boundary $\partial H(\Delta)$ in exactly one point of $\mathbb{C}\backslash\{0\}$.*

Let $\Sigma^*$ be the set of all harmonic, orientation-preserving, univalent mappings $f \in \Sigma$ which have starlike analytic functions $h+g$.

THEOREM 3.2. *If $f \in \Sigma^*$, then $\sum_{k=1}^{\infty} k|a_k + b_k|^2 \leq 1$.*

*Proof.* A starlike function $H(z) = h + g = z + \sum_{k=1}^{\infty}(a_k + b_k)z^{-k}$ is characterized by the condition

$$\frac{\partial}{\partial\theta}\{argH(re^{i\theta})\} > 0$$

for $r > 1$. But $argH(re^{i\theta}) = Im\{\log H(re^{i\theta})\}$, so that

$$\frac{\partial}{\partial\theta}(Im\{\log H(re^{i\theta})\}) = Im\left\{\frac{\partial}{\partial\theta}\log H(re^{i\theta})\right\} = Re\left\{\frac{zH'}{H}\right\} > 0.$$

From this, we have that

$$\left|\frac{1-\frac{zH'}{H}}{1+\frac{zH'}{H}}\right| < 1.$$

Thus

$$|H - zH'|^2 < |H + zH'|^2. \tag{3.1}$$

An integration of the left side of (3.1) gives

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} |H(re^{i\theta}) - re^{i\theta}H'(re^{i\theta})|^2 \, d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} (H(re^{i\theta}) - re^{i\theta}H'(re^{i\theta}))\overline{(H(re^{i\theta}) - re^{i\theta}H'(re^{i\theta}))} \, d\theta \\
&= \sum_{k=1}^{\infty}(k+1)^2|a_k + b_k|^2 r^{-2k}.
\end{aligned}$$

An integration of the right side of (3.1) gives

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}|H(re^{i\theta})+re^{i\theta}H'(re^{i\theta})|^2\,d\theta\\&=\frac{1}{2\pi}\int_0^{2\pi}(H(re^{i\theta})+re^{i\theta}H'(re^{i\theta}))\overline{(H(re^{i\theta})+re^{i\theta}H'(re^{i\theta}))}\,d\theta\\&=4r^2+\sum_{k=1}^{\infty}(1-k)^2|a_k+b_k|^2r^{-2k}.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{1}{2\pi}\int_0^{2\pi}|H(re^{i\theta})-re^{i\theta}H'(re^{i\theta})|^2\,d\theta\\&<\frac{1}{2\pi}\int_0^{2\pi}|H(re^{i\theta})+re^{i\theta}H'(re^{i\theta})|^2\,d\theta\end{aligned}$$

implies that

$$\sum_{k=1}^{\infty}(k+1)^2|a_k+b_k|^2r^{-2k}<4r^2+\sum_{k=1}^{\infty}(1-k)^2|a_k+b_k|^2r^{-2k}.$$

Simplify this, then we obtain

$$\sum_{k=1}^{\infty}4k|a_k+b_k|^2r^{-2k}<4r^2$$

for $r>1$. Letting $r\to 1$, we have that

$$\sum_{k=1}^{\infty}k|a_k+b_k|^2\le 1.$$

□

THEOREM 3.3. *If $f\in\Sigma^*$, then analytic function $H(z)=h(z)+g(z)$ is univalent.*

*Proof.* Let $G(\zeta)=\{H(1/\zeta)\}^{-1}$ for $|\zeta|<1$. Then

$$G(\zeta)=\zeta-(a_1+b_1)\zeta^3-(a_2+b_2)\zeta^4+\cdots$$

is analytic in $|\zeta|<1$ and satisfies that

$$Re\left\{\frac{\zeta G'(\zeta)}{G(\zeta)}\right\}=Re\left\{\frac{zH'(z)}{H(z)}\right\}>0. \tag{3.2}$$

If $G(\zeta_0) = 0$ at some point $0 < |\zeta_0| < 1$, then $\zeta[G'(\zeta)/G(\zeta)]$ has a simple pole at $\zeta_0$. This means that $Re\{\zeta[G'(\zeta)/G(\zeta)]\}$ takes on arbitrarily large negative values, contradicting to (3.2). Thus $G(\zeta)$ has no zeros in $|\zeta| < 1$ other than a simple zero at the origin. Let $0 < r < 1$. Since $G(\zeta)$ has one zero and no poles in $|\zeta| \leq r$, the argument principle tells us that $\Delta_{|\zeta|=r} argG(\zeta) = 2\pi$. That is, the circle $|\zeta| = r$ is mapped by $G(\zeta)$ onto a closed contour $C_r$ that winds around the origin once. Since $argG(\zeta)$ increases with $arg\zeta$, the curve cannot intersect itself. Hence $C_r$ is a simple closed contour. That is, $G(\zeta)$ is univalent on the circle $|\zeta| = r$ and therefore $G(\zeta)$ is univalent in $|\zeta| \leq r$. Since $r$ is arbitrary, the function $G(\zeta)$ is univalent in the unit disk $\mathbb{D} = \{\zeta : |\zeta| < 1\}$. This implies that $H(z)$ is univalent. □

THEOREM 3.4. *If $f \in \Sigma^*$, then $|a_n + b_n| \leq \frac{1}{\sqrt{n}}$.*

*Proof.* $f \in \Sigma^*$ implies that $H(z) = h + g = z + \sum_{k=1}^{\infty}(a_k + b_k)z^{-k}$ is univalent analytic function in $\Delta$ by Theorem 3.3. Thus we get $|a_1+b_1| \leq 1$ from [1] and $\sum_{k=1}^{\infty} k|a_k + b_k|^2 \leq 1$ from Theorem 3.2.

$$n|a_n + b_n|^2 \leq 1 - |a_1 + b_1|^2 \leq 1$$

for $n \geq 2$ and so $|a_n + b_n| \leq \frac{1}{\sqrt{n}}$. □

COROLLARY 3.5. *If $f \in \Sigma^*$ and $Re\{a_1 + b_1\} \leq \frac{nt^2-1}{nt^2+1}$ for $t > 0$, then*

$$Re\{t(a_1 + b_1) - (a_n + b_n)\} \leq t \quad \text{for} \quad n \geq 2.$$

*Proof.* In the proof of Theorem 3.4, we know that $n|a_n + b_n|^2 \leq 1 - |a_1 + b_1|^2 \leq 1$ for $n \geq 2$ and so $|a_n + b_n| \leq \frac{\sqrt{1-|a_1+b_1|^2}}{\sqrt{n}}$. Hence

$$\begin{aligned}
&Re\{t(a_1 + b_1) - (a_n + b_n)\} \\
&\leq tRe\{a_1 + b_1\} + \frac{1}{\sqrt{n}}\sqrt{1 - |a_1 + b_1|^2} \\
&\leq tRe\{a_1 + b_1\} + \frac{1}{\sqrt{n}}\sqrt{1 - (Re\{a_1 + b_1\})^2}.
\end{aligned}$$

Let $x = Re\{a_1 + b_1\}$, then $Re\{t(a_1 + b_1) - (a_n + b_n)\} \leq tx + \frac{1}{\sqrt{n}}\sqrt{1 - x^2}$. The function $F(x) = tx + \frac{1}{\sqrt{n}}\sqrt{1 - x^2}$ is increasing for $-1 \leq x \leq \frac{nt^2-1}{nt^2+1}$ and therefore $Re\{t(a_1 + b_1) - (a_n + b_n)\} \leq t$ for $n \geq 2$. □

COROLLARY 3.6. *If $f \in \Sigma^*$ and $Re\{a_1 + b_1\} \leq \frac{n^3-1}{n^3+1}$, then*

$$Re\{n(a_1 + b_1) - (a_n + b_n)\} \leq n.$$

*Proof.* Set $t = n$ in Corollary 3.5. □

COROLLARY 3.7. *If* $f \in \Sigma^*$, *then* $Re\{n(a_1 + b_1) - (a_n + b_n)\} \leq n$ *for all* $n$ *sufficiently large depending on* $f$.

*Proof.* Fix $f$. If $Re\{a_1 + b_1\} = 1$, then $a_n + b_n = 0$ for all $n \geq 2$ by Theorem 3.2 and the result holds for all $n \geq 2$. If $Re\{a_1 + b_1\} < 1$, then $Re\{a_1 + b_1\} \leq \frac{nn^2-1}{nn^2+1}$ for all $n$ sufficiently large since $(n^3-1)/(n^3+1) \to 1$ as $n \to \infty$. In this case the result follows from Corollary 3.6. □

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