Korean J. Math. **23** (2015), No. 4, pp. 619–630
http://dx.doi.org/10.11568/kjm.2015.23.4.619

MULTIPLICITY RESULT OF THE SOLUTIONS FOR A CLASS OF THE ELLIPTIC SYSTEMS WITH SUBCRITICAL SOBOLEV EXPONENTS

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Abstract. This paper is devoted to investigate the multiple solutions for a class of the cooperative elliptic system involving subcritical Sobolev exponents on the bounded domain with smooth boundary. We first show the uniqueness and the negativity of the solution for the linear system of the problem via the direct calculation. We next use the variational method and the mountain pass theorem in the critical point theory.

1. Introduction

Let Ω is a bounded domain of R^n with smooth boundary, $n \geq 3$, α, β, γ are real constants. In this paper we consider the multiplicity of the solutions for the following class of the cooperative elliptic system involving subcritical Sobolev exponents nonlinear term with Dirichlet boundary condition

$$-\Delta U = AU + \begin{pmatrix} F \\ G \end{pmatrix} \qquad \text{in } \ \Omega, \tag{1.1}$$

Received July 17, 2015. Revised December 2, 2015. Accepted December 3, 2015.
2010 Mathematics Subject Classification: 35J50, 35J55.
Key words and phrases: Cooperative elliptic system, subcritical Sobolev exponents nonlinear term, variational method, mountain pass theorem.
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† This work was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (KRF-2013010343).

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{on} \quad \partial\Omega.$$

Here $U = \binom{u}{v}$, $A = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$,

$$F = \frac{2p}{p+q} u_+^{p-1} v_+^q + f, \qquad G = \frac{2q}{p+q} u_+^p v_+^{q-1} + g,$$

where $u_+ = \max\{u, 0\}$, p, q are real constants, $2 < p+q < 2^*$, $2^* = \frac{2n}{n-2}$. We may write f, g as

$$f = t\phi_1 + f_1, \qquad g = s\phi_1 + g_1,$$

where ϕ_1 is the positive normalized function associated to the first eigenvalue λ_1 of the eigenvalue problem $-\Delta u = \lambda u$ in Ω, $u|_{\partial\Omega} = 0$, t, s are real constants, f_1, $g_1 \in L^2(\Omega)$ with

$$\int_\Omega f_1\phi_1 = \int_\Omega g_1\phi_1 = 0. \tag{1.2}$$

Our problems are characterized as Ambrosetti-Prodi type problems. Since the pioneering work on the subject in [2], these problem have been investigated in many ways. For a survey on the scalar case we recommend the paper [4] and the references therein. For the system case we recommend the paper [3]. Indeed the weak solutions of (1.1) correspond to the critical points of the continuous and *Frechét* differentiable functional

$$\begin{aligned} I(u,v) = & \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx \\ & - \int_\Omega [\frac{2}{p+q} u_+^p v_+^q + t\phi_1 u + f_1 u + s\phi_1 v + g_1 v]dx. \end{aligned} \tag{1.3}$$

Note that $2 < p+q < \frac{2n}{n-2}$ is the subcritical Sobolev exponents for the embedding $W_0^{1,2}(\Omega) \hookrightarrow L^{p+q}(\Omega)$, where $W_0^{1,2}(\Omega)$ is a Sobolev space. Since this embedding is compact(cf. [1]), the functional $I(u,v)$ satisfies the (PS) condition. Thus we can use the mountain pass theorem with (PS) condition to find the weak solution of (1.1).

Let $E = W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$ be a Hilbert space endowed with the norm

$$\|(u,v)\|_E^2 = \|u\|_{W_0^{1,2}(\Omega)}^2 + \|v\|_{W_0^{1,2}(\Omega)}^2.$$

Let A be $\begin{pmatrix} a & b \\ b & d \end{pmatrix} \in M_{2\times 2}(R)$ and $\mu^+_{\lambda_i}$ and $\mu^-_{\lambda_i}$ be the eigenvalues of the matrix
$\begin{pmatrix} \lambda_i - \alpha & -\beta \\ -\beta & \lambda_i - \gamma \end{pmatrix} \in M_{2\times 2}(R)$, i. e.,

$$\mu^+_{\lambda_i} = \frac{1}{2}\{-\gamma - \alpha + \sqrt{(-\gamma-\alpha)^2 - 4\{(\lambda_i - \alpha)(\lambda_i - \gamma) - \beta^2\}}\},$$

$$\mu^-_{\lambda_i} = \frac{1}{2}\{-\gamma - \alpha - \sqrt{((-\gamma-\alpha))^2 - 4\{(\lambda_i - \alpha)(\lambda_i - \gamma) - \beta^2\}}\}.$$

We note that

$$\text{if } \; 4\{(\lambda_i - \alpha)(\lambda_i - \gamma) - \beta^2\} < 0, \text{ then } \mu^-_{\lambda_i} < 0 < \mu^+_{\lambda_i},$$

$$\text{if } \; -\gamma \geq \alpha \text{ and } 4\{(\lambda_i - \alpha)(\lambda_i - \gamma) - \beta^2\} > 0, \text{ then } 0 < \mu^-_{\lambda_i} < \mu^+_{\lambda_i}.$$

$$\text{if } \; -\gamma \leq \alpha \text{ and } 4\{(\lambda_i - \alpha)(\lambda_i - \gamma) - \beta^2\} > 0, \text{ then } \mu^-_{\lambda_i} < \mu^+_{\lambda_i} < 0.$$

We are looking for the weak solutions of (1.1) in E. The weak solutions in E satisfies

$$\int_\Omega [(-\Delta u, -\Delta v)\cdot(z,w) - (\alpha u + \beta v, \beta u + \gamma v)\cdot(z,w) \\ - (t\phi_1 + f_1, s\phi_1 + g_1)\cdot(z,w)]dx = 0 \;\; \forall (z,w) \in E.$$

Our main result is as follows:

Theorem 1.1. *Assume that*

(i) *det* $\begin{pmatrix} \lambda_i - \alpha & -\beta \\ -\beta & \lambda_i - \gamma \end{pmatrix} > 0$ *for* $i \geq 1$,

(ii) $\alpha > 0$, $\beta > 0$, $\gamma < 0$, $\lambda_1 - \alpha > 0$.

Then there exists (t_1, s_1) *with* $t_1 < 0$ *and* $s_1 < 0$ *such that for any* (t, s) *with* $t < t_1$ *and* $s < s_1$, *(1.1) has at least two weak solutions* (u, v), *one of which is a negative solution.*

In section 2, we obtain a negative solution of (1.1) by direct computation. In section 3, we approach the variational technique and show the existence of the second weak solution of (1.1) by the mountain pass theorem with (PS) condition, so we prove Theorem 1.1.

2. A negative solution

Lemma 2.1. *Assume that the conditions* (i) *and* (ii) *of Theorem 1.1 hold. Let* $M_{\alpha\beta\gamma} : E \to E$ *be the operator defined by* $M_{\alpha\beta\gamma}(u, v) = (-\Delta u - \alpha u - \beta v, -\Delta v - \beta u - \gamma v)$. *Then the operator*

$$M_{\alpha\beta\gamma}^{-1} : E \to E$$

is well defined and continuous, and the system

$$\begin{cases} -\Delta u &= \alpha u + \beta v + f_1 \quad \text{in } \Omega, \\ -\Delta v &= \beta u + \gamma v + g_1, \quad \text{in } \Omega, \\ \quad u &= v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.1}$$

has a unique solution (u_0, v_0), *which is of the form*

$$u_0 = \sum_m \left(\frac{(\lambda_m - \gamma)h_m + \beta k_m}{(\lambda_m - \alpha)(\lambda_m - \gamma) - \beta^2}\right)\phi_m,$$

$$v_0 = \sum_m \left(\frac{(\lambda_m - \alpha)k_m + \beta h_m}{(\lambda_m - \alpha)(\lambda_m - \gamma) - \beta^2}\right)\phi_m,$$

where $f_1 = \sum_m h_m\phi_m$ *with* $\sum_m h_m^2 < +\infty$ *and* $g_1 = \sum_m k_m\phi_m$ *with* $\sum_m k_m^2 < +\infty$.

Proof. Let us take (f_1, g_1) in E. Then we can write $f_1 = \sum_m h_m\phi_m$ with $\sum_m h_m^2 < +\infty$ and $g_1 = \sum_m k_m\phi_m$ with $\sum_m k_m^2 < +\infty$. We define, for m integers,

$$u_m = \frac{(\lambda_m - \gamma)h_m + \beta k_m}{(\lambda_m - \alpha)(\lambda_m - \gamma) - \beta^2}, \qquad v_m = \frac{(\lambda_m - \alpha)k_m + \beta h_m}{(\lambda_m - \alpha)(\lambda_m - \gamma) - \beta^2}, \tag{2.2}$$

which make sense since $(\lambda_m - \alpha)(\lambda_m - \gamma) - \beta^2 \neq 0$ for every m. We note that

$$|u_m| \leq \frac{C}{|\lambda_m|}(|h_m| + |k_m|),$$

from which it follows that

$$\lambda_m^2 u_m^2 \leq C_1(h_m^2 + k_m^2)$$

for suitable constants C, C_1 not depending on m. We apply the same inequality for v_m. So if $u_0 = \sum_m u_m\phi_m$, $v_0 = \sum_m v_m\phi_m$, then $(u_0, v_0) \in E$. We can check easily that

$$M_{\alpha\beta\gamma}(u_0, v_0) = (f_1, g_1).$$

So $M^{-1}_{\alpha\beta\gamma}: E \to E$ is well defined, so we prove the lemma. □

The following Lemma 2.2 come from Lemma 2.1.

LEMMA 2.2. *Assume that the conditions* (i) *and* (ii) *of Theorem 1.1 hold. Then for any (t,s) with* $t < 0$ *and* $s < 0$, *the linear system*

$$\begin{cases} -\Delta u &= \alpha u + \beta v + t\phi_1 \quad \text{in } \Omega, \\ -\Delta v &= \beta u + \gamma v + s\phi_1 \quad \text{in } \Omega, \\ \quad u &= v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.3}$$

has a unique negative solution $(u_*, v_*) \in E$, *which is of the form*

$$u_* = [\frac{\beta^2 s + \beta s(\lambda_1 - \alpha)}{(\lambda_1 - \alpha)((\lambda_1 - \alpha)(\lambda_1 - \gamma) - \beta^2)} + \frac{t}{\lambda_1 - \alpha}]\phi_1 < 0,$$

$$v_* = [\frac{\beta t + s(\lambda_1 - \alpha)}{(\lambda_1 - \alpha)(\lambda_1 - \gamma) - \beta^2}]\phi_1 < 0.$$

Proof. We note that (u_*, v_*) is a solution of system (2.3) for any (t, s) with $t < 0$ and $s < 0$, and the uniqueness is the consequence of Lemma 2.1. □

The following lemma can be obtained by Lemma 2.1 and Lemma 2.2.

LEMMA 2.3. *Assume that the conditions* (i) *and* (ii) *of Theorem 1.1 hold. Then there exist constants* $t_* < 0$ *and* $s_* < 0$ *such that for any* (t, s) *with* $t < t_*$ *and* $s < s_*$, *the system*

$$\begin{cases} -\Delta u &= \alpha u + \beta v + t\phi_1 + f_1 \quad \text{in } \Omega, \\ -\Delta v &= \beta u + \gamma v + s\phi_1 + g_1 \quad \text{in } \Omega, \\ \quad u &= v = 0 \quad \text{on } \partial\Omega \end{cases} \tag{2.4}$$

has a unique negative solution $(u_{ts}, v_{ts}) \in E$ *with* $u_{ts} < 0$ *and* $v_{ts} < 0$, *which is the negative solution of (1.1) and of the form*

$$\begin{aligned} u_{ts} &= u_0 + u_* \\ &= \sum_m (\frac{(\lambda_m - \gamma)h_m + \beta k_m}{(\lambda_m - \alpha)(\lambda_m - \gamma) - \beta^2})\phi_m \\ &\quad + [\frac{\beta^2 t + \beta s(\lambda_1 - \alpha)}{(\lambda_1 - \alpha)((\lambda_1 - \alpha)(\lambda_1 - \gamma) - \beta^2)} + \frac{t}{\lambda_1 - \alpha}]\phi_1, \end{aligned}$$

$$\begin{aligned} v_{ts} &= v_0 + v_* \\ &= \sum_m (\frac{(\lambda_m - \alpha)k_m + \beta h_m}{(\lambda_m - \alpha)(\lambda_m - \gamma) - \beta^2})\phi_m + [\frac{\beta t + s(\lambda_1 - \alpha)}{(\lambda_1 - \alpha)(\lambda_1 - \gamma) - \beta^2}]\phi_1, \end{aligned}$$

where $f_1 = \sum_m h_m\phi_m$ *with* $\sum_m h_m^2 < +\infty$ *and* $g_1 = \sum_m k_m\phi_m$ *with* $\sum_m k_m^2 < +\infty$.

Proof. Since for any (t, s) with $t < 0$ and $s < 0$, $u_* < 0$ and $v_* < 0$, we can choose $t^* < 0$ and $s^* < 0$ such that for any (t, s) with $t < t_*$ and $s < s_*$, $u_{ts} = u_0 + u_* < 0$ and $v_{ts} = v_0 + v_* < 0$. □

3. Second solution and Proof of Theorem 1.1

We observe that the weak solutions of (1.1) coincide with the critical points of the the associated functional

$$I : E \to R \in C^{1,1},$$

$$I(u, v) = Q_{\alpha\beta\gamma}(u, v) - \int_\Omega [\frac{2}{p+q}u_+^p v_+^q + t\phi_1 u + f_1 u + s\phi_1 v + g_1 v]dx, \quad (3.1)$$

where

$$Q_{\alpha\beta\gamma}(u, v) = \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx.$$

We note that if (u, v) is a solution of (1.1), then $(u, v) = (z, w) + (u_{ts}, v_{ts})$, where (u_{ts}, v_{ts}) is a negative solution of (1.1) and (z, w) is a nontrivial solution of the system

$$\begin{cases} -\Delta u &= \alpha u + \beta v + \frac{2p}{p+q}(u + u_{ts})_+^{p-1}(v + v_{ts})_+^q \quad \text{in } \Omega, \\ -\Delta v &= \beta u + \gamma v + \frac{2q}{p+q}(u + u_{ts})_+^{p}(v + v_{ts})_+^{q-1} \quad \text{in } \Omega, \\ \quad u &= v = 0 \quad \text{on } \partial\Omega. \end{cases} \quad (3.2)$$

Thus it suffices to find the nontrivial solution of (3.2) to find the solution of (1.1). We observe that the weak solutions of (3.2) coincide with the critical points of the functional

$$F : E \to R \in C^{1,1},$$

$$F(u, v) = \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx$$
$$- \frac{2}{p+q}\int_\Omega (u + u_{ts})_+^p (v + v_{ts})_+^q dx. \quad (3.3)$$

Thus it suffices to find the critical points of F. Let us set

$$H_{\lambda_i} = \text{span}\{\phi_j |\ \lambda_j = \lambda_i\}.$$

Let us denote by $(c_{\lambda_i}^+, d_{\lambda_i}^+)$ and $(c_{\lambda_i}^-, d_{\lambda_i}^-)$ the eigenvectors of

$\begin{pmatrix} \lambda_i - \alpha & -\beta \\ -\beta & \lambda_i - \gamma \end{pmatrix} \in M_{2\times 2}(R)$ corresponding to $\mu^+_{\lambda_i}$ and $\mu^-_{\lambda_i}$ respectively. Let us set

$$\begin{aligned} D_{\lambda_i} &= \{(\alpha,\beta,\gamma)\in R^3|\ (\lambda_i-\alpha)(\lambda_i-\gamma)-\beta^2\geq 0\},\\ D'_{\lambda_i} &= D_{\lambda_i}\cap\{-\gamma\leq\alpha\},\\ D''_{\lambda_i} &= D_{\lambda_i}\cap\{-\gamma\geq\alpha\},\\ E_{\lambda_i} &= \{(c\phi,d\phi)\in E|\ (c,d)\in R^2,\phi\in H_{\lambda_i}\},\\ E^+_{\lambda_i} &= \{(c^+_{\lambda_i}\phi,d^+_{\lambda_i}\phi)\in E|\ \phi\in H_{\lambda_i}\},\\ E^-_{\lambda_i} &= \{(c^-_{\lambda_i}\phi,d^-_{\lambda_i}\phi)\in E|\ \phi\in H_{\lambda_i}\},\\ H^+(\alpha,\beta,\gamma) &= (\oplus_{\mu^+_{\lambda_i}>0}E^+_{\lambda_i})\oplus(\oplus_{\mu^-_{\lambda_i}>0}E^-_{\lambda_i}),\\ H^-(\alpha,\beta,\gamma) &= (\oplus_{\mu^+_{\lambda_i}<0}E^+_{\lambda_i})\oplus(\oplus_{\mu^-_{\lambda_i}<0}E^-_{\lambda_i}),\\ H^0(\alpha,\beta,\gamma) &= (\oplus_{\mu^+_{\lambda_i}=0}E^+_{\lambda_i})\oplus(\oplus_{\mu^-_{\lambda_i}=0}E^-_{\lambda_i}). \end{aligned}$$

Then $H^+(\alpha,\beta,\gamma)$, $H^-(\alpha,\beta,\gamma)$ and $H^0(\alpha,\beta,\gamma)$ are the positive, negative and null space relative to the quadratic form $Q_{\alpha,\beta,\gamma}$ in E. Because $(\lambda_i - \alpha)(\lambda_i-\gamma)-b^2\neq 0$,

$$H^0(\alpha,\beta,\gamma)=\{0\}.$$

LEMMA 3.1. *Assume that the conditions* (i) *and* (ii) *of Theorem 1.1 hold. Let* $(\alpha,\beta,\gamma)\in R^3$. *Then*
(i) $E^+_{\lambda_i}$ *and* $E^-_{\lambda_i}$ *are eigenspace for the operator* $M_{\alpha\beta\gamma}$, $M_{\alpha\beta\gamma}(u,v) = (-\Delta u-\alpha u-\beta v,-\Delta v-\beta u-\gamma v)$ *associated with* $Q_{\alpha\beta\gamma}$ *with eigenvalues* $\frac{\mu^+_{\lambda i}}{\lambda_i}$ *and* $\frac{\mu^-_{\lambda_i}}{\lambda_i}$ *respectively.*
(ii) $E^+_{\lambda_i}$ *and* $E^-_{\lambda_i}$ *generate* E.
(iii) Let $i\geq 1$. *Then we have that*

$$\text{if}\ \ (\alpha,\beta,\gamma)\in D'_{\lambda_i},\ \ \mu^-_{\lambda_i}<\mu^+_{\lambda_i}\leq 0,$$

$$\text{if}\ \ (\alpha,\beta,\gamma)\in D''_{\lambda_i},\ \ 0\leq\mu^-_{\lambda_i}<\mu^+_{\lambda_i},$$

$$\lim_{(\alpha,\beta,\gamma)\to(\alpha_0,\beta_0,\gamma_0)}\mu^-_{\lambda_i}(\alpha,\beta,\gamma)=\mu^-_{\lambda_i}(\alpha_0,\beta_0,\gamma_0)$$

and

$$\lim_{(\alpha,\beta,\gamma)\to(\alpha_0,\beta_0,\gamma_0)}\mu^+_{\lambda_i}(\alpha,\beta,\gamma)=\mu^+_{\lambda_i}(\alpha_0,\beta_0,\gamma_0).$$

uniformly with respect to $i\in N$.

Proof. The proof can be obtained by easy computations. □

Let us define

$$C_{p,q}(\Omega) = \inf_{(u,v)\in E\setminus(0,0)} \frac{\int_\Omega(|\nabla u|^2 + |\nabla v|^2)dx}{(\int_\Omega |u|^p|v|^q dx)^{\frac{2}{p+q}}} \quad \text{for } (u,v)\in E. \tag{3.4}$$

$$C_{p+q}(\Omega) = \inf_{(u,v)\in E\setminus\{(0,0)}\} \frac{\int_\Omega(|\nabla u|^2)dx}{(\int_\Omega |u|^{p+q})^{\frac{2}{p+q}}} \quad \text{for } u\in W_0^{1,2}(\Omega). \tag{3.5}$$

LEMMA 3.2. *Let Ω be a domain (not necessarily bounded) and $\alpha+\beta \le 2^*$. Then we have*

$$C_{p,q}(\Omega) = [(\frac{p}{q})^{\frac{q}{p+q}} + (\frac{p}{q})^{\frac{-p}{p+q}}]C_{p+q}(\Omega). \tag{3.6}$$

Proof. The proof can be found in [1]. □

We shall show that F satisfies the mountain pass geometry.

Let $(\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}$. Let W be any neighborhood of $(\alpha_0,\beta_0,\gamma_0)$. Then

$$W = (W\cap(\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}} D'_{\lambda_i}))\oplus(W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}} D'_{\lambda_i}).$$

Thus we have that

$$\text{if } (\alpha,\beta,\gamma)\in W\cap(\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}} D'_{\lambda_i}), \text{ then } \mu^-_{\lambda_i} < \mu^+_{\lambda_i} < 0 \qquad \forall i\ge 1 \tag{3.7}$$

and

$$\text{if } (\alpha,\beta,\gamma)\in W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}} D'_{\lambda_i}, \text{ then } 0<\mu^-_{\lambda_i} < \mu^+_{\lambda_i} \qquad \forall i\ge 1. \tag{3.8}$$

By (3.8), we have that if $(\alpha,\beta,\gamma)\in W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}} D'_{\lambda_i}$, then $E = H^+(\alpha,\beta,\gamma)$.

Let $(\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}$ and W be a neighborhoodof $(\alpha_0,\beta_0,\gamma_0)$. Then by Lemma 3.1, we have, for any $(\alpha,\beta,\gamma)\in W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}} D'_{\lambda_i}$ and $(u,v)\in E$,

$$\begin{aligned} Q_{\alpha\beta\gamma}(u,v) &= \frac{1}{2}\int_\Omega[|\nabla u)|^2 + |\nabla v|^2 - \alpha u^2 - 2\beta uv - \gamma v^2]dx \\ &> a\|(u,v)\|^2_{L^2(\Omega)} > 0 \qquad \text{for some } a>0. \end{aligned} \tag{3.9}$$

We note that $F(0,0)=0$.

LEMMA 3.3. *Assume that the conditions (i) and (ii) of Theorem 1.1 hold. Let $i\in N$ and $(\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}$. Then there exist a neighborhood W of $(\alpha_0,\beta_0,\gamma_0)$ such that for any $(\alpha,\beta,\gamma)\in W\setminus\cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}}$*

D'_{λ_i},
(i) there exists a constant $\rho > 0$ *such that*

$$F(u,v) > 0 \ \ \forall U \in \partial B_\rho, \qquad F(u,v) > -\infty \ \ \forall U \in B_\rho,$$

where B_ρ *is a ball centered at (0,0) with radius* $\rho > 0$*, and*
(ii) there exist a constant $R > 0$ *and an element* $U_0 \in E$ *such that*

$$F(U_0) < 0 \qquad \text{for } \ \|U_0\| > R, \qquad F(u,v) < \infty \ \ \forall (u,v) \in B_R,$$

where B_R *is a ball centered at (0,0) with radius* $R > 0$*.*

Proof. (i) Let (α,β,γ) be any element of $W \setminus \cup_{i\in N,\ (\alpha_0,\beta_0,\gamma_0)\in\partial D'_{\lambda_i}} D'_{\lambda_i}$. By (3.4) and (3.9), we have

$$\begin{aligned} F(u,v) &= Q_{\alpha\beta\gamma}(u,v) - \frac{2}{p+q}\int_\Omega (u+u_{ts})_+^p (v+v_{ts})_+^q dx \\ &> a\|(u,v)\|^2_{L^2(\Omega)} - \frac{2}{p+q}\int_\Omega |u|^p |v|_+^q dx \\ &> a\|(u,v)\|^2_{L^2(\Omega)} - \frac{2}{p+q}(C_{p,q}(\Omega))^{-\frac{p+q}{2}}\|(u,v)\|_E^{p+q}. \end{aligned}$$

Since $2 < p+q < \frac{2n}{n-2}$, there exists a small constant $\rho > 0$ such that if $(u,v) \in \partial B_\rho$, then $F(u,v) > 0$. Moreover if $(u,v) \in B_\rho$, then $F(u,v) \geq -\frac{2}{p+q}(C_{p,q}(\Omega))^{-\frac{p+q}{2}}\|(u,v)\|_E^{p+q} > -\infty$.
(ii) Let us choose an element $(e_1,e_2) \in E$ with $(e_1,e_2) \neq (0,0)$ such that

$$\int_\Omega (e_1-1)_+^p (e_2-1)_+^q dx > 0. \tag{3.10}$$

Let $\sigma > 0$ be any real number. Then we have

$$F(\sigma(e_1,e_2)) = \sigma^2 Q_{\alpha\beta\gamma}(u,v) - \sigma^{p+q}\frac{2}{p+q}\int_\Omega (e_1 + \frac{u_{ts}}{\sigma})_+^p (e_2 + \frac{v_{ts}}{\sigma})_+^q dx.$$

If we choose $\sigma_1 > 0$ such that $\frac{u_{ts}(x)}{\sigma_1} \geq -1$, $\frac{v_{ts}(x)}{\sigma_1} \geq -1$, $\forall x \in \Omega$, then we have, by (3.10), that $\int_\Omega (e_1 + \frac{u_{ts}}{\sigma})_+^p (e_2 + \frac{v_{ts}}{\sigma})_+^q dx > 0$ for any $\sigma \geq \sigma_1$, it follows from that

$$F(\sigma(e_1,e_2)) \to -\infty \qquad \text{as} \quad \sigma \to \infty.$$

Thus there exist $\sigma' > 0$ and a constant $R > 0$ such that $F(\sigma'(e_1,e_2)) < 0$ and $\|\sigma'(e_1,e_2)\| > R$. Then $U_0 = \sigma'(e_1,e_2)$ is the required point such that $F(U_0) < 0$ and $\|U_0\| > R$. Moreover if $(u,v) \in B_R$, then $F(u,v) = Q_{\alpha\beta\gamma}(u,v) - \frac{2}{p+q}\int_\Omega (u+u_{ts})_+^p (v+v_{ts})_+^q dx \leq Q_{\alpha\beta\gamma}(u,v) < +\infty$. □

LEMMA 3.4. *Assume that the conditions (i) and (ii) of Theorem 1.1 hold. Then the functional F satisfies the $(P.S.)_c$ condition for any any real number c.*

Proof. Let $i \in N$, $(\alpha_0, \beta_0, \gamma_0) \in \partial D'_{\lambda_i}$ and W be a neighborhood of $(\alpha_0, \beta_0, \gamma_0)$. Let $(\alpha, \beta, \gamma) \in W \setminus \cup_{i \in N,\ (\alpha_0,\beta_0,\gamma_0) \in \partial D'_{\lambda_i}} D'_{\lambda_i}$. Let $c \in R$ and (h_n) be a sequence in N such that $h_n \to +\infty$, $(u_n, v_n)_n$ be a sequence such that $(u_n, v_n) \in E_{h_n}$, $\forall n$, $F(u_n, v_n) \to c$ and $DF(u_n, v_n) \to \theta$, $\theta = (0, \cdots, 0)$. We claim that $(u_n, v_n)_n$ is bounded. By contradiction we suppose that $\|(u_n, v_n)\|_E \to +\infty$ and set $(\hat{u_n}, \hat{v_n}) = \frac{(u_n,v_n)}{\|(u_n,v_n)\|_E}$. If $(\alpha, \beta, \gamma) \in W \setminus \cup_{i \in N,\ (\alpha_0,\beta_0,\gamma_0) \in \partial D'_{\lambda_i}} D'_{\lambda_i}$, then $Q_{\alpha\beta\gamma}(u_n, v_n) > 0$, it follows that we have

$$c \leftarrow F(u_n, v_n) = Q_{\alpha\beta\gamma}(u_n, v_n) - \frac{2}{p+q} \int_\Omega (u_n + u_{ts})_+^p (v_n + v_{ts})_+^q dx$$
$$> -\frac{2}{p+q} \int_\Omega (u_n + u_{ts})_+^p (v_n + v_{ts})_+^q dx.$$

Thus

$$-\frac{2}{p+q} \int_\Omega (u_n + u_{ts})_+^p (v_n + v_{ts})_+^q dx \qquad \text{is bounded} \quad .$$

We also have

$$DF(u_n, v_n) \cdot (\hat{u_n}, \hat{v_n}) = 2\frac{F(u_n, v_n)}{\|(u_n, v_n)\|_E} -$$
$$\frac{\int_\Omega (\frac{2p}{p+q}(u_n + u_{ts})_+^{p-1}(v_n + v_{ts})_+^q u_n + \frac{2q}{p+q}(u_n + u_{ts})_+^p (v_n + v_{ts})_+^{q-1} v_n}{\|(u_n, v_n)\|_E}$$
$$\frac{-\frac{4}{p+q}(u_n + u_{ts})_+^p (v_n + v_{ts})_+^q) dx}{\|(u_n, v_n)\|_E} \longrightarrow 0.$$

It follows from $F(u_n, v_n)$ is bounded and $\|(u_n, v_n)\|_E \to \infty$ that

$$2\frac{F(u_n, v_n)}{\|(u_n, v_n)\|_E} \to 0$$

and

$$\frac{\int_\Omega (\frac{2p}{p+q}(u_n + u_{ts})_+^{p-1}(v_n + v_{ts})_+^q u_n + \frac{2q}{p+q}(u_n + u_{ts})_+^\alpha (v_n + v_{ts})_+^{q-1} v_n) dx}{\|(u_n, v_n)\|_E}$$
$$-\frac{\frac{4}{p+q}\int_\Omega (u_n + u_{ts})_+^p (v_n + v_{ts})_+^q dx}{\|(u_n, v_n)\|_E} \longrightarrow 0.$$

Since $-\frac{4}{p+q}\int_\Omega (u_n+u_{ts})_+^p(v_n+v_{ts})_+^q dx$ is bounded in Ω,

$$\frac{-\frac{4}{p+q}\int_\Omega (u_n+u_{ts})_+^p(v_n+v_{ts})_+^q dx}{\|(u_n,v_n)\|_E} \text{ converges to } 0$$

and

$$\frac{\int_\Omega \operatorname{grad}(\frac{2}{p+q}(u_n+u_{ts})_+^p(v_n+v_{ts})_+^q dx)\cdot(u_n,v_n)dx}{\|(u_n,v_n)\|_E} \text{ converges to } 0.$$

We note that

$$\frac{DF(u_n,v_n)}{\|(u_n,v_n)\|_E^2} = M_{\alpha,\beta,\gamma}(\hat{u_n},\hat{v_n}))_n - \frac{\operatorname{grad}(\frac{2}{p+q}(u_n+u_{ts})_+^p(v_n+v_{ts})_+^q)}{\|(u_n,v_n)\|_E^2} \longrightarrow \theta,$$

where $M_{\alpha\beta\gamma}(u,v)=(-\Delta u-\alpha u-\beta v,-\Delta v-\beta u-\gamma v)$. Since $\frac{\operatorname{grad}(\frac{2}{p+q}(u_n+u_{ts})_+^p(v_n+v_{ts})_+^q)}{\|(u_n,v_n)\|_E^2}$ converges to θ, $\theta=(0,0)$, $(M_{\alpha,\beta,\gamma}(\hat{u_n},\hat{v_n}))_n$ converges to θ. Since $(\hat{u_n},\hat{v_n})_n$ is bounded and $M^{-1}_{\alpha,\beta,\gamma}$ is a compact mapping, up to subsequence, $(\hat{u_n},\hat{v_n})_n$ converges strongly to $M^{-1}_{\alpha\beta\gamma}(\theta)=\theta$, which is a contradiction to the fact that $\|(\hat{u_n},\hat{v_n})\|_E=1$. Thus $(u_n,v_n)_n$ is bounded. Since $2<p+q<\frac{2n}{n-2}$, the embedding $W_0^{1,2}(\Omega)\hookrightarrow L^{p+q}(\Omega)$, is compact, by Lemma (3.2), the sequence (u_n,v_n) has a subsequence, up to a subsequence, (u_n,v_n) which converges strongly to some $(u_0.v_0)$ with $DF(u_0,v_0)=\lim DF(u_n,v_n)=0$. Thus we prove the lemma. □

Proof of Theorem 1.1

We note that the functional $F\in C^1(E,R)$ and $F(0,0)=0$. By Lemma 3.4, F satisfies $(PS)_c$ condition for any real number. Let us define

$$\Gamma=\{\gamma\in C([0,1],E)|\ \gamma(0)=(0,0),\quad \gamma(1)=U_0\},$$

where U_0 is a point in E such that $F(U_0)<0$. Let us define

$$\tau = inf_{\gamma\in\Gamma}\sup_{(u,v)\in\gamma(t)} F(u,v).$$

By Lemma 3.3, there exist a constant $\rho>0$ and an element U_0 such that $F|_{\partial B_\rho}>0$ and $F(U_0)<0$. By the mountain pass theorem (cf. [5]), τ is a critical value of F with a critical point (u_1,v_1) such that

$$\tau = F(u_1,v_1).$$

Thus we prove Theorem 1.1.

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