Korean J. Math. **23** (2015), No. 4, pp. 665–732
http://dx.doi.org/10.11568/kjm.2015.23.4.665

STUDIES ON BOUNDARY VALUE PROBLEMS FOR BILATERAL DIFFERENCE SYSTEMS WITH ONE-DIMENSIONAL LAPLACIANS

Xiaohui Yang and Yuji Liu*

Abstract. Existence results for multiple positive solutions of two classes of boundary value problems for bilateral difference systems are established by using a fixed point theorem under convenient assumptions. It is the purpose of this paper to show that the approach to get positive solutions of boundary value problems of finite difference equations by using multi-fixed-point theorems can be extended to treat the bilateral difference systems with one-dimensional Laplacians. As an application, the sufficient conditions are established for finding multiple positive homoclinic solutions of a bilateral difference system. The methods used in this paper may be useful for numerical simulation. An example is presented to illustrate the main theorems. Further studies are proposed at the end of the paper.

1. Introduction

Difference equations appear naturally as analogues and as numerical solutions of differential and delay differential equations having applications in applied digital control, biology, ecology, economics, physics and so on [37]. It is extremely difficult to understand thoroughly the behaviors of their solution. Recent studies on finite difference equations

Received April 27, 2015. Revised November 2, 2015. Accepted December 1, 2015.
2010 Mathematics Subject Classification: 39A11, 34B10, 34B15.
Key words and phrases: One-dimension Laplacian, bilateral difference system, boundary value problem, positive solution, homoclinic solution, fixed point theorem.
∗ Corresponding author.

may be seen in [2, 4, 5, 9, 10, 18, 20–22, 24, 35, 38, 39, 52, 55, 63, 64] and the references therein.

Boundary value problems for difference equations can be studied in several ways: usually, numerical analysis is employed together with fixed point methods or other tools from nonlinear operator theory (see the classical monographs of Agarwal [1], Kelley and Peterson [33] and Lakshmikanham and Trigiante [34]). In the last decade, the use of variational methods in such problems has gained increasing interest: this was made possible by Agarwal, Perera and O'Regan [8], who established a convenient variational framework for discrete BVP's, analogous to that used in differential equations.

Many authors have applied different results in critical point theory to prove existence and multiplicity results for the solutions of discrete BVP's: let us mention the works of Cabada, Iannizzotto and Tersian [14], Cai, Guo and Yu [16], Candito and Giovannelli [17], Faraci and Iannizzotto [23], Jiang and Zhou [30], Kong [31], Long [36], Mihailescu, Radulescu and Tersian [54], and Ricceri [57] and so on. We also mention the papers [25, 26, 42, 45, 49, 51, 58, 61] concerning the existence of periodic solutions, homoclinic solutions, subharmonic solutions of bilateral difference equations and papers [40, 43, 44, 46–48, 50, 59, 60, 65] concerning the solvability of boundary value problems of difference equations on bounded discrete intervals of the type $[m, n] = \{k \in \mathbf{Z} : m \leq k \leq n\}$ which allows one to search for solutions in a finite-dimensional Banach space.

The asymptotic theory of difference equations is an area in which there is great activity among a large number of investigators. In this theory, it is of great interest to investigate, in particular, the existence of solutions with prescribed asymptotic behavior, which are global in the sense that they are solutions on the whole discrete interval ($\mathbf{Z} = \{0, \pm 1, \pm 2, \cdots\}$) or semi-infinite discrete intervals ($\mathbb{N} = \{0, 1, 2, \cdots\}$). The existence of global solutions with prescribed asymptotic behavior is usually formulated as the existence of solutions of boundary value problems on the whole discrete interval ($\mathbf{Z} = \{0, \pm 1, \pm 2, \cdots\}$) or semi-infinite discrete intervals ($\mathbb{N} = \{0, 1, 2, \cdots\}$). The issue of finding solutions for discrete BVP's on unbounded discrete intervals is more delicate.

Cabada and Cid [12] and Cabada and Tersian [13] and authors in [3, 6, 7, 32, 62] studied the existence of solutions of difference equations defined semi-infinite discrete intervals ($\mathbb{N} = \{0, 1, 2, \cdots\}$) by using different methods such as the approximation method.

In [62], the existence of multiple positive solutions of the boundary value problems for second-order discrete equations

(1)
$$\begin{cases} \Delta^2 x(n-1) - p\Delta x(n-1) - qx(n-1) + f(n, x(n)) = 0, n \in \mathbb{N}, \\ \alpha x(0) - \beta \Delta x(0) = 0, \quad \lim\limits_{n\to+\infty} x(n) = 0, \end{cases}$$

was investigated by using the cone compression and expansion fixed point theorems in Banach spaces, where $\alpha > 0, \beta > 0$, $p > 0, q > 0$ and f is a continuous function.

In paper [3], it was considered the existence of solutions of a class of the infinite time scale boundary value problems. It is easy to see that the results in [3] can be applied to the following BVP for the infinite difference equation

$$(2) \qquad \begin{cases} \Delta^2 x(n) + f(n, x(n)) = 0, \quad n \in \mathbb{N}, \\ x(0) = 0, \quad x(n) \text{ is bounded.} \end{cases}$$

The methods used in [3] are based upon the growth argument and the upper and lower solutions methods.

In [56], the authors dealt with the second-order non-autonomous difference equation

$$(3) \qquad \Delta x(n) = \left(\frac{n}{n+1}\right)^2 \left(\Delta x(n-1) + h^2 f(x(n))\right), n \in \mathbb{N},$$

which is a difference model reduced from a differential model in hydrodynamics or in nonlinear field theory. (3) can be transformed to the following form

$$\Delta(n^2 \Delta x(n-1)) = h^2 (n+1)^2 f(x(n)), n \in \mathbb{N},$$

where $h > 0$ is a parameter and f is Lipschitz continuous and has three real zeros $L_0 < 0 < L$, conditions for f under which for each sufficiently small $h > 0$ there exists a homoclinic solution of the above equation were presented.

In [66], the periodic difference equation with saturable nonlinearity defined on whole discrete interval ($\mathbf{Z} = \{0, \pm 1, \pm 2, \cdots\}$)

$$(4)\quad a_n x(n+1) + a_{n-1}x(n-1) + b_n x(n) - \omega x(n) = \frac{\sigma \chi_n x(n)^3}{1 + c_n x(n)^2}, n \in \mathbf{Z}$$

was considered, where Z denotes the set of all integers, $\{a_n\}$ and $\{b_n\}$ are real valued $T-$periodic sequences, $\{\chi_n\}$ and $\{c_n\}$ are positive real valued $T-$periodic sequences. It is easy to see that (4) can be changed to the following form

$$\Delta(a_{n-1}\Delta x(n-1)) = \frac{\sigma \chi_n x(n)^3}{1 + c_n x(n)^2} + [\omega - b_n - a_n - a_{n-1}]x(n), n \in \mathbf{Z}.$$

In [37], authors investigated the existence at least three solutions of the following boundary value problem of the second order bilateral difference system

$$\begin{cases} \Delta[p(n)\phi(\Delta x(n))] + f(n, x(n), y(n)) = 0, \quad n \in \mathbf{Z}, \\ \Delta[q(n)\psi(\Delta y(n))] + g(n, x(n), y(n)) = 0, \quad n \in \mathbf{Z}, \\ \lim\limits_{n \to -\infty} x(n) - \sum\limits_{n=-\infty}^{+\infty} \alpha_n x(n) = \lim\limits_{n \to -\infty} y(n) - \sum\limits_{n=-\infty}^{+\infty} \gamma_n y(n) = 0, \\ \lim\limits_{n \to +\infty} \phi^{-1}(p(n))\Delta x(n) - \sum\limits_{n=-\infty}^{+\infty} \beta_n \Delta x(n) = 0, \\ \lim\limits_{n \to +\infty} \psi^{-1}(q(n))\Delta y(n) - \sum\limits_{n=-\infty}^{+\infty} \delta_n \Delta y(n) = 0. \end{cases}$$

In applications, it occurs that there exist many kinds of bilateral difference systems [1, 2, 11, 28, 41] and the nonlinearities always depend on the Δ operator. Motivated by [28, 29, 37], the purpose of this paper is to investigate the following boundary value problems of the second order bilateral difference systems with one-dimensional Laplacians

$$(5)\quad \begin{cases} \Delta[p(n)\Phi(\Delta x(n))] + f(n, y(n), \Delta y(n)) = 0, & n \in \mathbf{Z}, \\ \Delta[q(n)\Psi(\Delta y(n))] + g(n, x(n), \Delta x(n)) = 0, & n \in \mathbf{Z}, \\ \lim\limits_{n\to-\infty} x(n) - \sum\limits_{n=-\infty}^{+\infty} \alpha_n x(n) = \lim\limits_{n\to-\infty} y(n) - \sum\limits_{n=-\infty}^{+\infty} \gamma_n y(n) = 0, \\ \lim\limits_{n\to+\infty} x(n) - \sum\limits_{n=-\infty}^{+\infty} \beta_n x(n) = \lim\limits_{n\to+\infty} y(n) - \sum\limits_{n=-\infty}^{+\infty} \delta_n y(n) = 0 \end{cases}$$

and

$$(5')\quad \begin{cases} \Delta[p(n)\Phi(\Delta x(n))] + f(n, y(n), \Delta y(n)) = 0, & n \in \mathbf{Z}, \\ \Delta[q(n)\Psi(\Delta y(n))] + g(n, x(n), \Delta x(n)) = 0, & n \in \mathbf{Z}, \\ \lim\limits_{n\to-\infty} x(n) - \sum\limits_{n=-\infty}^{+\infty} \alpha_n x(n) = \lim\limits_{n\to-\infty} y(n) - \sum\limits_{n=-\infty}^{+\infty} \gamma_n y(n) = 0, \\ \lim\limits_{n\to+\infty} \frac{x(n)}{1+\sum\limits_{s=-\infty}^{n} \frac{1}{\Phi^{-1}(p(s))}} - \sum\limits_{n=-\infty}^{+\infty} \beta_n x(n) = 0, \\ \lim\limits_{n\to+\infty} \frac{y(n)}{1+\sum\limits_{s=-\infty}^{n} \frac{1}{\Psi^{-1}(q(s))}} - \sum\limits_{n=-\infty}^{+\infty} \delta_n y(n) = 0 \end{cases}$$

where

(a) $\mathbf{Z}$ denotes the set of all integers, $\Delta x(n) = x(n+1) - x(n)$, $\Delta y(n) = y(n+1) - y(n)$,

(b) $p(n), q(n) > 0$ for all $n \in \mathbf{Z}$ satisfying

$$\sum_{s=0}^{+\infty} \frac{1}{\Phi^{-1}(p(s))}, \sum_{s=-\infty}^{0} \frac{1}{\Phi^{-1}(p(s))}, \sum_{s=0}^{+\infty} \frac{1}{\Psi^{-1}(q(s))}, \sum_{s=-\infty}^{0} \frac{1}{\Psi^{-1}(q(s))} < +\infty,$$

(b′) $p(n), q(n) > 0$ for all $n \in \mathbf{Z}$ satisfying

$$\sum_{s=0}^{+\infty} \frac{1}{\Phi^{-1}(p(s))} = \sum_{s=0}^{+\infty} \frac{1}{\Psi^{-1}(q(s))} = +\infty, \sum_{s=-\infty}^{0} \frac{1}{\Phi^{-1}(p(s))}, \sum_{s=-\infty}^{0} \frac{1}{\Psi^{-1}(q(s))} < +\infty,$$

(c) $\alpha_n, \beta_n, \gamma_n, \delta_n \geq 0$ for all $n \in \mathbf{Z}$ and satisfy

$$\sum_{n=-\infty}^{+\infty} \alpha_n < 1, \quad \sum_{n=-\infty}^{+\infty} \beta_n < 1, \quad \sum_{i=-\infty}^{+\infty} \alpha_i \sum_{s=-\infty}^{i-1} \frac{1}{\Phi^{-1}(p(s))} < +\infty,$$

$$\sum_{n=-\infty}^{+\infty} \gamma_n < 1, \quad \sum_{n=-\infty}^{+\infty} \delta_n < 1, \quad \sum_{i=-\infty}^{+\infty} \gamma_i \sum_{s=-\infty}^{i-1} \frac{1}{\Psi^{-1}(q(s))} < +\infty,$$

(c′) $\alpha_n, \beta_n, \gamma_n, \delta_n \geq 0$ for all $n \in \mathbf{Z}$ and satisfy

$$\sum_{n=-\infty}^{+\infty} \alpha_n < 1, \quad \sum_{n=-\infty}^{+\infty} \gamma_n < 1$$

$$1 - \frac{\sum_{n=-\infty}^{+\infty} \beta_n}{1 - \sum_{n=-\infty}^{+\infty} \alpha_n} \sum_{n=-\infty}^{+\infty} \alpha_n \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} - \sum_{n=-\infty}^{+\infty} \beta_n \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} > 0,$$

$$1 - \frac{\sum_{n=-\infty}^{+\infty} \delta_n}{1 - \sum_{n=-\infty}^{+\infty} \gamma_n} \sum_{n=-\infty}^{+\infty} \gamma_n \sum_{t=-\infty}^{n-1} \frac{1}{\Psi^{-1}(q(t))} - \sum_{n=-\infty}^{+\infty} \delta_n \sum_{t=-\infty}^{n-1} \frac{1}{\Psi^{-1}(q(t))} > 0,$$

(d) $f, g : \mathbf{Z} \times [0, +\infty) \times \mathbb{R} \to [0, +\infty)$, f is a $q-$Carathéodory function, and g a $p-$Carathéodory function (see Definitions 9 and 10 in Section 2), and $f(n, 0, 0)^2 + g(n, 0, 0)^2 > 0$ for $n \in \mathbf{Z}$,

(d′) $f, g : \mathbf{Z} \times [0, +\infty) \times \mathbb{R} \to [0, +\infty)$, f is a $q-$sub-Carathéodory function, and g a $p-$sub-Carathéodory function (see Definitions 5 and 6 in Section 2), and $f(n, 0, 0)^2 + g(n, 0, 0)^2 > 0$ for $n \in \mathbf{Z}$,

(e) Φ is defined by $\Phi(x) = |x|^{s-2}x$ with $s > 1$, and $\Psi(x) = |x|^{t-2}x$ with $t > 1$ are called one-dimensional Laplacian or Laplacian operator, their inverse functions are denoted by Φ^{-1} and Ψ^{-1} respectively, Φ, Ψ are called one-dimensional Laplacian operators.

A pair of bilateral sequences $\{(x(n), y(n))\}$ is called a solution of BVP(5) (or BVP(5′)) if $x(n), y(n)$ satisfy all equations in (5) (or (5′)). A solution $\{(x(n), y(n))\}$ of BVP(5) (or BVP(5′)) is called a positive solution of (5) (or (5′)) if $x(n) \geq 0, y(n) \geq 0$ for all $n \in \mathbf{Z}$ and any integers $M > m$, $x(n)^2 + y(n)^2 \not\equiv 0$ for $n \in [m, M]$.

We establish sufficient conditions for the existence of at least three positive solutions of BVP(5) and BVP(5′) respectively. This paper may be the first one to study the solvability boundary value problems of bilateral difference systems with $\sum\limits_{n=-\infty}^{+\infty}\frac{1}{\Phi^{-1}(p(n))}=\sum\limits_{n=-\infty}^{+\infty}\frac{1}{\Psi^{-1}(q(n))}=+\infty$. The most interesting part in this article is to transform these BVPs to operator equations and to construct nonlinear operators on cones of suitable Banach spaces, this constructing method is not found in known papers.

As an application, we consider the existence of multiple positive solutions of the following problem:

$$
(6)\qquad \begin{cases} \Delta[(|n|+1)^2\Delta x(n)]+f(n,y(n),\Delta y(n))=0, & n\in \mathbf{Z},\\ \Delta[(|n|+1)^2\Delta y(n)]+g(n,x(n),\Delta x(n))=0, & n\in \mathbf{Z},\\ \lim\limits_{n\to-\infty}x(n)=0, \quad \lim\limits_{n\to-\infty}y(n)=0, & \\ \lim\limits_{n\to+\infty}x(n)=0, \quad \lim\limits_{n\to+\infty}y(n)=0, & \end{cases}
$$

where $f,g:\mathbf{Z}\times[0,+\infty)\times\mathbb{R}\to[0,+\infty)$ are continuous functions satisfying for each $n_0\in Z$, $f(n,0,0)^2+g(n,0,0)^2\not\equiv 0$ for $n\le n_0$, for each $n_1\in\mathbf{Z}$, $f(n,0,0)^2+g(n,0,0)^2\not\equiv 0$ for $n\ge n_1$. Obviously, (6) is a special case of (5) with $\phi(x)=\psi(x)=x$, $p(n)=q(n)=(|n|+1)^2$, $\alpha_n=\beta_n=\gamma_n=\delta_n=0$ for all $n\in\mathbf{Z}$.

As usual, we say that a solution (x,y) of the following bilateral difference system

$$
(7)\qquad \begin{cases} \Delta[(|n|+1)^2\Delta x(n)]+f(n,y(n),\Delta y(n))=0, & n\in \mathbf{Z},\\ \Delta[(|n|+1)^2\Delta y(n)]+g(n,x(n),\Delta x(n))=0, & n\in \mathbf{Z}, \end{cases}
$$

is homoclinic (to 0) if

$$
\lim_{|n|\to+\infty}x(n)=\lim_{|n|\to+\infty}y(n)=0 \text{ holds.}
$$

Hence the solutions of (6) are homoclinic solutions of (7). Thus the sufficient conditions are established for finding multiple positive homoclinic solutions of the bilateral difference system (7).

The remainder of this paper is organized as follows: in Section 2, we first give some technical lemmas and preliminary results. The first result is proved for the existence of multiple positive solutions of BVP(5) under (a), (b), (c), (d) and (e) in Section 3. The second result for the existence of solutions of BVP(5′) under (a), (b′), (c′), (d′) and (e) is proved in Section 4. An example is presented in Section 5.

2. Preliminary Results

In this section, we present some background definitions in Banach spaces, state an important three fixed point theorem [9] and then prove some technical lemmas.

Definition 1. [9] Let E be a real Banach space. The nonempty convex closed subset P of E is called a cone in E if $ax \in P$ for all $x \in P$ and $a \geq 0$, $x \in E$ and $-x \in E$ imply $x = 0$.

Definition 2. [9] A map $\varphi : P \to [0, +\infty)$ is a nonnegative continuous concave or convex functional map provided φ is nonnegative, continuous and satisfies $\varphi(tx + (1-t)y) \geq t\varphi(x) + (1-t)\varphi(y)$, or $\varphi(tx + (1-t)y) \leq t\varphi(x) + (1-t)\varphi(y)$, for all $x, y \in P$ and $t \in [0, 1]$.

Definition 3. [9] An operator $T : E \to E$ is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 4. [9] Let $a, b, c, d, h > 0$ be positive constants, α, φ be two nonnegative continuous concave functionals on the cone P, γ, β, θ be three nonnegative continuous convex functionals on the cone P. Define the convex sets as follows:

$$
\begin{aligned}
&P_c = \{x \in P : ||x|| < c\},\\
&P(\gamma, \alpha; a, c) = \{x \in P : \alpha(x) \geq a,\ \gamma(x) \leq c\},\\
&P(\gamma, \theta, \alpha; a, b, c) = \{x \in P : \alpha(x) \geq a,\ \theta(x) \leq b,\ \gamma(x) \leq c\},\\
&Q(\gamma, \beta; , d, c) = \{x \in P : \beta(x) \leq d,\ \gamma(x) \leq c\},\\
&Q(\gamma, \beta, \varphi; h, d, c) = \{x \in P : \varphi(x) \geq h,\ \beta(x) \leq d,\ \gamma(x) \leq c\}.
\end{aligned}
$$

Lemma 1. [9, 10] *Let E be a real Banach space, P be a cone in E, α, φ be two nonnegative continuous concave functionals on the cone P, γ, β, θ be three nonnegative continuous convex functionals on the cone P. There exist constant $M > 0$ such that*

$$\alpha(x) \leq \beta(x),\ ||x|| \leq M\gamma(x) \text{ for all } x \in P.$$

Furthermore, Suppose that $h, d, a, b, c > 0$ are constants with $d < a$. Let $T : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator. If

(C1): $\{y \in P(\gamma, \theta, \alpha; a, b, c) | \alpha(x) > a\} \neq \emptyset$ *and*

$$\alpha(Tx) > a \text{ for every } x \in P(\gamma, \theta, \alpha; a, b, c);$$

(C2): $\{y \in Q(\gamma, \theta, \varphi; h, d, c) | \beta(x) < d\} \neq \emptyset$ *and*

$$\beta(Tx) < d \text{ for every } x \in Q(\gamma, \theta, \varphi; h, d, c);$$

(C3): $\alpha(Ty) > a$ *for* $y \in P(\gamma, \alpha; a, c)$ *with* $\theta(Ty) > b$;

(C4): $\beta(Tx) < d$ *for each* $x \in Q(\gamma, \beta; d, c)$ *with* $\varphi(Tx) < h$,

then T *has at least three fixed points* y_1, y_2 *and* y_3 *such that*

$$\beta(y_1) < d,\ \alpha(y_2) > a,\ \beta(y_3) > d,\ \alpha(y_3) < a.$$

Definition 5. f is called a $q-$sub-Carathéodory function if it satisfies that

$$(u,v) \to f\left(n, \left(1 + \sum_{-\infty}^{n} \frac{1}{\Psi^{-1}(q(n))}\right) u, \frac{1}{\Psi^{-1}(q(n))} v\right)$$

is continuous, and for each $r > 0$ there exists a nonnegative bilateral real number sequence $\{\phi_r(n)\}$ with $\sum_{n=-\infty}^{+\infty} \phi_r(n) < +\infty$ such that

$$\left| f\left(n, \left(1 + \sum_{-\infty}^{n} \frac{1}{\Psi^{-1}(q(n))}\right) u, \frac{1}{\Psi^{-1}(q(n))} v\right) \right| \leq \phi_r(n), n \in \mathbf{Z}, |u| \leq r, |v| \leq r.$$

Definition 6. f is called a $q-$Carathéodory function if it satisfies that

$$(u,v) \to f\left(n, u, \frac{1}{\Psi^{-1}(q(n))} v\right)$$

is continuous, and for each $r > 0$ there exists a nonnegative bilateral real number sequence $\{\phi_r(n)\}$ with $\sum_{n=-\infty}^{+\infty} \phi_r(n) < +\infty$ such that

$$\left| f\left(n, u, \frac{1}{\Psi^{-1}(q(n))} v\right) \right| \leq \phi_r(n), n \in \mathbf{Z}, |u| \leq r, |v| \leq r.$$

3. Existence of positive solutions of BVP(5′)

In this section, we establish existence result for three positive solutions of BVP(5′).

Choose

$$X=\left\{\{x(n)\}:\begin{array}{c} x(n)\in \mathbb{R}, n\in \mathbf{Z},\text{ there exist the limits}\\ \lim\limits_{n\to+\infty}\frac{x(n)}{1+\sum\limits_{s=-\infty}^{n}\frac{1}{\Phi^{-1}(p(s))}},\ \lim\limits_{n\to-\infty}x(n),\\ \lim\limits_{n\to+\infty}\Phi^{-1}(p(n))\Delta x(n),\ \lim\limits_{n\to-\infty}\Phi^{-1}(p(n))\Delta x(n)\end{array}\right\}.$$

Define the norm

$$||x||_X=\max\left\{\sup_{n\in\mathbf{Z}}\frac{|x(n)|}{1+\sum\limits_{s=-\infty}^{n}\frac{1}{\Phi^{-1}(p(s))}},\ \sup_{n\in\mathbf{Z}}\Phi^{-1}(p(n))|\Delta x(n)|\right\},x\in X.$$

It is easy to see that X is a real Banach space.

Choose

$$Y=\left\{\{y(n)\}:\begin{array}{c} y(n)\in \mathbb{R}, n\in \mathbf{Z},\text{ there exist the limits}\\ \lim\limits_{n\to+\infty}\frac{y(n)}{1+\sum\limits_{s=-\infty}^{n}\frac{1}{\Psi^{-1}(q(s))}},\ \lim\limits_{n\to-\infty}y(n),\\ \lim\limits_{n\to+\infty}\Psi^{-1}(q(n))\Delta y(n),\ \lim\limits_{n\to-\infty}\Psi^{-1}(q(n))\Delta y(n)\end{array}\right\}.$$

Define the norm

$$||y||_Y=\max\left\{\sup_{n\in\mathbf{Z}}\frac{|y(n)|}{1+\sum\limits_{s=-\infty}^{n}\frac{1}{\Psi^{-1}(q(s))}},\ \sup_{n\in\mathbf{Z}}\Psi^{-1}(q(n))|\Delta y(n)|\right\},y\in Y.$$

It is easy to see that Y is a real Banach space.

Let $E=X\times Y$ be defined by $E=\{(x,y):x\in X,y\in Y\}$ with the norm $||(x,y)||=\max\{||x||,||y||\}$. Then E is a Banach space.

Denote

$$\varepsilon_1=\frac{\sum\limits_{n=-\infty}^{+\infty}\beta_n}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}+\sum_{n=-\infty}^{+\infty}\beta_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))},$$

$$\varepsilon_2=\frac{\sum\limits_{n=-\infty}^{+\infty}\delta_n}{1-\sum\limits_{n=-\infty}^{+\infty}\gamma_n}\sum_{n=-\infty}^{+\infty}\gamma_n\sum_{t=-\infty}^{n-1}\frac{1}{\Psi^{-1}(q(t))}+\sum_{n=-\infty}^{+\infty}\delta_n\sum_{t=-\infty}^{n-1}\frac{1}{\Psi^{-1}(q(t))},$$

$$\epsilon_1=\frac{\Phi(\varepsilon_1)}{1-\Phi(\varepsilon_1)},\ \epsilon_2=\frac{\Psi(\varepsilon_2)}{1-\Psi(\varepsilon_2)}.$$

Lemma 2. *Suppose that* (b'), (c') *and* (e) *hold and* $h(n) \not\equiv 0 (n \in \mathbf{Z})$ *be a nonnegative sequence with* $\sum\limits_{n=-\infty}^{+\infty} h(n)$ *converging. Then there exists a unique number* $A_h \in \left[0, \epsilon_1 \sum\limits_{n=-\infty}^{+\infty} h(n)\right]$ *such that*

$$\begin{aligned}\Phi^{-1}(A_h) &= \frac{\sum\limits_{n=-\infty}^{+\infty}\beta_n}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n} \sum_{n=-\infty}^{+\infty} \alpha_n \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} \Phi^{-1}\left(A_h + \sum_{s=t}^{+\infty} h(s)\right) \\ &+ \sum_{n=-\infty}^{+\infty} \beta_n \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} \Phi^{-1}\left(A_h + \sum_{s=t}^{+\infty} h(s)\right).\end{aligned} \tag{8}$$

Proof. Let

$$\begin{aligned}G(c) = 1 &- \frac{\sum\limits_{n=-\infty}^{+\infty}\beta_n}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n} \sum_{n=-\infty}^{+\infty} \alpha_n \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} \Phi^{-1}\left(1 + \frac{1}{c}\sum_{s=t}^{+\infty} h(s)\right) \\ &- \sum_{n=-\infty}^{+\infty} \beta_n \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} \Phi^{-1}\left(1 + \frac{1}{c}\sum_{s=t}^{+\infty} h(s)\right).\end{aligned}$$

It is easy to see that G is increasing on $(-\infty, 0)$ and $(0, +\infty)$ respectively. One has

$$\lim_{c\to 0^-} G(c) = +\infty, \ \lim_{c\to 0^+} G(c) = -\infty,$$

$$\begin{aligned}\lim_{c\to -\infty} G(c) = 1 &- \frac{\sum\limits_{n=-\infty}^{+\infty}\beta_n}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n} \sum_{n=-\infty}^{+\infty} \alpha_n \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} \\ &- \sum_{n=-\infty}^{+\infty} \beta_n \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} > 0.\end{aligned}$$

Furthermore, we have

$$
\begin{aligned}
&G\left(\epsilon_1 \sum_{n=-\infty}^{+\infty} h(n)\right)\\
&= 1 - \frac{\sum_{n=-\infty}^{+\infty}\beta_n}{1-\sum_{n=-\infty}^{+\infty}\alpha_n} \sum_{n=-\infty}^{+\infty}\alpha_n \sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(1+\frac{1}{\epsilon_1\sum_{n=-\infty}^{+\infty}h(n)}\sum_{s=t}^{+\infty}h(n)\right)\\
&- \sum_{n=-\infty}^{+\infty}\beta_n \sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(1+\frac{1}{\epsilon_1\sum_{n=-\infty}^{+\infty}h(n)}\sum_{s=t}^{+\infty}h(n)\right)\\
&\geq 1 - \frac{\sum_{n=-\infty}^{+\infty}\beta_n}{1-\sum_{n=-\infty}^{+\infty}\alpha_n} \sum_{n=-\infty}^{+\infty}\alpha_n \sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(1+\frac{1}{\epsilon_1}\right)\\
&- \sum_{n=-\infty}^{+\infty}\beta_n \sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(1+\frac{1}{\epsilon_1}\right) = 1-\varepsilon_1\Phi^{-1}\left(1+\frac{1}{\epsilon_1}\right)=0.
\end{aligned}
$$

Hence there exists a unique $A_h \in \left[0, \epsilon_1 \sum_{n=-\infty}^{+\infty} h(n)\right]$ such that (8) holds. The proof is complete. □

LEMMA 3. *Suppose that (b′), (c′) and (e) hold and $h(n) \not\equiv 0 (n \in \mathbf{Z})$ be a nonnegative sequence with $\sum_{n=-\infty}^{+\infty} h(n)$ converging. Then there exists a unique number $B_h \in \left[0, \epsilon_2 \sum_{n=-\infty}^{+\infty} h(n)\right]$ such that*

$$
\begin{aligned}
\Psi^{-1}(B_h) &= \frac{\sum_{n=-\infty}^{+\infty}\delta_n}{1-\sum_{n=-\infty}^{+\infty}\gamma_n} \sum_{n=-\infty}^{+\infty}\gamma_n \sum_{t=-\infty}^{n-1}\frac{1}{\Psi^{-1}(q(t))}\Psi^{-1}\left(B_h+\sum_{s=t}^{+\infty}h(s)\right)\\
&+ \sum_{n=-\infty}^{+\infty}\delta_n \sum_{t=-\infty}^{n-1}\frac{1}{\Psi^{-1}(q(t))}\Psi^{-1}\left(B_h+\sum_{s=t}^{+\infty}h(s)\right).
\end{aligned} \tag{9}
$$

Proof. The proof is similar to Lemma 1 and is omitted. □

Consider the following BVP

$$
(10)\qquad \begin{cases} \Delta[p(n)\Phi(\Delta x(n))] + h(n) = 0, \quad n \in \mathbf{Z}, \\ \lim\limits_{n\to-\infty} x(n) - \sum\limits_{n=-\infty}^{+\infty} \alpha_n x(n) = 0, \\ \lim\limits_{n\to+\infty} \frac{x(n)}{1+\sum\limits_{n=-\infty}^{n} \frac{1}{\Phi^{-1}(p(n))}} - \sum\limits_{n=-\infty}^{+\infty} \beta_n x(n) = 0, \end{cases}
$$

Lemma 4. *Suppose that (b′), (c′) and (e) hold. Then x is a nonnegative solution of BVP(10) if and only if*

$$
(11)\qquad \begin{aligned} x(n) = {} & \frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n} \sum_{n=-\infty}^{+\infty} \alpha_n \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} \Phi^{-1}\left(A_h + \sum_{s=t}^{+\infty} h(s)\right) \\ & + \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} \Phi^{-1}\left(A_h + \sum_{s=t}^{+\infty} h(s)\right), \end{aligned}
$$

where $A_h \in \left[0, \epsilon_1 \sum\limits_{s=-\infty}^{+\infty} h(s)\right]$ *satisfying (8).*

Proof. Suppose that x is a solution of (10). Because $\sum\limits_{n=-\infty}^{+\infty} h(n)$ is convergent, so there exists $A \in \mathbb{R}$ such that $p(n)\Phi(\Delta x(n)) = A + \sum\limits_{s=n}^{+\infty} h(s)$. Thus there exists $B \in \mathbb{R}$ such that

$$
x(n) = B + \sum_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} \Phi^{-1}\left(A + \sum_{s=t}^{+\infty} h(s)\right).
$$

Since $\sum\limits_{n=-\infty}^{n} \frac{1}{\Phi^{-1}(p(n))} \to +\infty$ as $n \to +\infty$, then

$$
\lim_{n\to+\infty} \frac{x(n)}{1+\sum\limits_{n=-\infty}^{n} \frac{1}{\Phi^{-1}(p(n))}} = \lim_{n\to+\infty} \Phi^{-1}(p(n))\Delta x(n).
$$

Thus we have $\lim\limits_{n\to+\infty}\Phi^{-1}(p(n))\Delta x(n)=\sum\limits_{n=-\infty}^{+\infty}\beta_n x(n)$. Hence the boundary conditions in (10) imply that

$$\Phi^{-1}(A)=B\sum_{n=-\infty}^{+\infty}\beta_n+\sum_{n=-\infty}^{+\infty}\beta_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A+\sum_{s=t}^{+\infty}h(s)\right),$$

$$B=B\sum_{n=-\infty}^{+\infty}\alpha_n+\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A+\sum_{s=t}^{+\infty}h(s)\right).$$

It follows that

$$B=\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A+\sum_{s=t}^{+\infty}h(s)\right),$$

$$\Phi^{-1}(A)=\frac{\sum\limits_{n=-\infty}^{+\infty}\beta_n}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A+\sum_{s=t}^{+\infty}h(s)\right)$$

$$+\sum_{n=-\infty}^{+\infty}\beta_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A+\sum_{s=t}^{+\infty}h(s)\right).$$

From Lemma 2, we know that $A=A_h$ and $A_h\in\left[0,\epsilon_1\sum\limits_{n=-\infty}^{+\infty}h(n)\right]$ satisfies (8). Hence we get (11).

Now suppose that x satisfies (11) and A_h satisfies (8). It is easy to show that x is a solution of (10). We need to prove that x is positive. From $\Delta[p(n)\Phi(\Delta x(n))]=-h(n)\le 0$ for all $n\in\mathbf{Z}$, we see $\Phi^{-1}(p(n))\Delta x(n)$ is decreasing. Since $A_h=\lim\limits_{n\to+\infty}p(n)\Phi(\Delta x(n))\ge 0$, then $\Phi^{-1}(p(n))\Delta x(n)\ge 0$ for all n. So $x(n)$ is increasing. Then $\lim\limits_{n\to-\infty}x(n)=\sum\limits_{n=-\infty}^{+\infty}\alpha_n x(n)\ge\lim\limits_{n\to-\infty}x(n)\sum\limits_{n=-\infty}^{+\infty}\alpha_n$. So

$$\left(1-\sum_{n=-\infty}^{+\infty}\alpha_n\right)\lim_{n\to-\infty}x(n)\ge 0.$$

By (c′), we know $\lim\limits_{n\to-\infty}x(n)\ge 0$. Then $x(n)\ge 0$ for all $n\in\mathbf{Z}$. The proof is complete. □

Consider the following BVP

$$
(12)\qquad \begin{cases} \Delta[q(n)\Psi(\Delta y(n))] + h(n) = 0, \quad n \in \mathbf{Z}, \\ \lim\limits_{n\to-\infty} y(n) - \sum\limits_{n=-\infty}^{+\infty} \gamma_n y(n) = 0, \\ \lim\limits_{n\to+\infty} \frac{y(n)}{1+\sum\limits_{n=-\infty}^{n} \frac{1}{\Psi^{-1}(q(n))}} - \sum\limits_{n=-\infty}^{+\infty} \delta_n y(n) = 0, \end{cases}
$$

Similarly to Lemma 4, we can prove the following lemma.

LEMMA 5. *Suppose that (b′), (c′) and (e) hold. Then x is a nonnegative solution of BVP(12) if and only if*

$$
(13)\qquad \begin{aligned} y(n) &= \frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\gamma_n} \sum_{n=-\infty}^{+\infty} \gamma_n \sum_{t=-\infty}^{n-1} \frac{1}{\Psi^{-1}(q(t))} \Phi^{-1}\left(B_h + \sum_{s=t}^{+\infty} h(s)\right) \\ &+ \sum_{t=-\infty}^{n-1} \frac{1}{\Psi^{-1}(q(t))} \Psi^{-1}\left(B_h + \sum_{s=t}^{+\infty} h(s)\right), \end{aligned}
$$

where $B_h \in \left[0, \epsilon_2 \sum\limits_{s=-\infty}^{+\infty} h(s)\right]$ *satisfying (9).*

Choose $k \in \mathbf{Z}$ with $k > 2$. Denote

$$
\begin{aligned} &P_n = 1 + \sum_{s=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(s))}, Q_n = 1 + \sum_{s=-\infty}^{n-1} \frac{1}{\Psi^{-1}(q(s))}, \\ &P(t) = P_n + (P_{n+1} - P_n)(t-n),\ t \in [n, n+1], n \in \mathbf{Z}, \\ &Q(t) = Q_n + (Q_{n+1} - Q_n)(t-n),\ t \in [n, n+1], n \in \mathbf{Z}. \end{aligned}
$$

Then both P and Q are strictly increasing on $\mathbb{R}$. Let $t = P^{-1}(\tau)$ and $t = Q^{-1}(\tau)$ be the inverse function of $\tau = P(t) - 1$ and $\tau = Q(t) - 1$ respectively.

LEMMA 6. *Suppose that (b′), (c′) and (e) hold, $h(n) \geq 0$ for all $n \in \mathbf{Z}$. Suppose x is a solution of BVP(10). Then*

$$
(14)\qquad \min_{n\in[-k,k]} \frac{x(n)}{P_n} \geq \mu_1 \sup_{n\in Z} \frac{x(n)}{P_n},
$$

where

$$
(15)\qquad \mu_1 = \frac{P_{-k} - 1}{P_k}.
$$

Proof. Since $\Delta[p(n)\Phi(\Delta x(n))] = -h(n) \leq 0$ for all $n \in \mathbf{Z}$, we see that $p(n)\Phi(\Delta x(n))$ is decreasing. Then $\Phi^{-1}(p(n))\Delta x(n)$ is decreasing.

It follow from Lemma 4 that $\Phi^{-1}(p(n))\Delta x(n) \geq 0$, $x(n)$ is increasing and $x(n) \geq 0$ for all $n \in \mathbf{Z}$. Denote $x(t) = x(n)+(x(n+1)-x(n))(t-n)$ for all $t \in [n, n+1], n \in Z$. Then $x'(t) = \Delta x(n)$ for all $t \in (n, n+1)$. So

$$\Delta x(n) = x'(t) = \frac{dx}{d\tau}\frac{d\tau}{dt} = \frac{dx}{d\tau}\frac{dP(t)}{dt} = \frac{dx}{d\tau}\Delta P_n = \frac{1}{\Phi^{-1}(p(n))}\frac{dx}{d\tau}.$$

Then $\Phi^{-1}(p(n))\Delta x(n) = \frac{dx}{d\tau}$. Hence x is concave with respect to τ.

Suppose that there exists $n_0 \in \mathbf{Z}$ such that $\sup_{n\in\mathbf{Z}} \frac{x(n)}{P_n} = \frac{x(n_0)}{P_{n_0}}$. Hence

$$\begin{aligned}
&\min_{n\in[-k,k]} \frac{x(n)}{P_n} \geq \frac{x(-k)}{P_k} = \frac{x(t(\tau(-k)))}{P_k} = \frac{x(t(P_{-k}-1))}{P_k} \\
&= \frac{1}{P_k} x\left(t\left(\frac{P_{n_0}-P_{-k}-1}{P_{n_0}}\frac{P_{-k}-1}{P_{n_0}-P_{-k}-1} + \frac{P_{-k}-1}{P_{n_0}}[P_{n_0}-1]\right)\right) \\
&\geq \frac{\frac{P_{n_0}-P_{-k}-1}{P_{n_0}} x\left(t\left(\frac{P_{-k}-1}{P_{n_0}-P_{-k}-1}\right)\right) + \frac{(P_{-k}-1)x\left(t\left(P_{n_0}-1\right)\right)}{P_{n_0}}}{P_k} \\
&\geq \frac{1}{P_k}\frac{P_{-k}-1}{P_{n_0}} x\left(t\left(P_{n_0}-1\right)\right) \\
&= \frac{1}{P_k}\frac{P_{-k}-1}{P_{n_0}} x\left(n_0\right) \geq \frac{P_{-k}-1}{P_k}\frac{x\left(n_0\right)}{P_{n_0}} \geq \mu_1 \sup_{n\in\mathbf{Z}} \frac{x(n)}{P_n}.
\end{aligned}$$

If $\sup_{n\in\mathbf{Z}} \frac{x(n)}{P_n} = \lim_{n\to\pm\infty} \frac{x(n)}{P_n}$, we choose $n_0 \in \mathbf{Z}$. Similarly to above discussion, we get

$$\min_{n\in[-k,k]} \frac{x(n)}{P_n} \geq \mu_1 \frac{x(n_0)}{P_{n_0}}.$$

Let $n \to \pm\infty$, we get (14). The proof is complete. □

LEMMA 7. *Suppose that (b′), (c′) and (e) hold, $h(n) \geq 0$ for all $n \in \mathbf{Z}$. Suppose y is a solution of BVP(12). Then*

$$\min_{n\in[-k,k]} \frac{y(n)}{Q_n} \geq \mu_2 \sup_{n\in Z} \frac{y(n)}{Q_n}, \tag{16}$$

where

$$\mu_2 = \frac{Q_{-k}-1}{Q_k}. \tag{17}$$

Proof. The proof is similar to Lemma 6 and is omitted. □

Let $\mu=\min\{\mu_1,\mu_2\}$. Define the cone P in $X\times Y=E$ by

$$P=\left\{(x,y)\in E:\begin{array}{l} x(n),y(n)\geq 0, n\in \mathbf{Z},\\ \lim\limits_{n\to-\infty} x(n)-\sum\limits_{n=-\infty}^{+\infty}\alpha_n x(n)=0,\\ \lim\limits_{n\to+\infty}\Phi^{-1}(p(n))\Delta x(n)-\sum\limits_{n=-\infty}^{+\infty}\beta_n x(n)=0,\\ \lim\limits_{n\to-\infty} y(n)-\sum\limits_{n=-\infty}^{+\infty}\gamma_n y(n)=0,\\ \lim\limits_{n\to+\infty}\Psi^{-1}(q(n))\Delta y(n)-\sum\limits_{n=-\infty}^{+\infty}\delta_n y(n)=0,\\ \min\limits_{n\in[-k,k]}\frac{x(n)}{P_n}\geq \mu\sup\limits_{n\in\mathbf{Z}}\frac{x(n)}{P_n},\\ \min\limits_{n\in[-k,k]}\frac{y(n)}{Q_n}\geq \mu\sup\limits_{n\in\mathbf{Z}}\frac{y(n)}{Q_n}\end{array}\right\}.$$

For $(x,y)\in P$, define $(T(x,y))(n)=((T_1y)(n),(T_2x)(n))$ by

$$(T_1y)(n)=$$

$$\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A_f(y)+\sum_{s=t}^{+\infty}f(s,y(s),\Delta y(s))\right)$$

$$+\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A_f(y)+\sum_{s=t}^{+\infty}f(s,y(s),\Delta y(s))\right),$$

where $A_f(y)\in\left[0,\epsilon_1\sum\limits_{s=-\infty}^{+\infty}f(s,y(s),\Delta y(s))\right]$ satisfying

$$\Phi^{-1}(A_f(y))=$$

$$\frac{\sum\limits_{n=-\infty}^{+\infty}\beta_n}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A_f(y)+\sum_{s=t}^{+\infty}f(s,y(s),\Delta y(s))\right)$$

$$+\sum_{n=-\infty}^{+\infty}\beta_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A_f(y)+\sum_{s=t}^{+\infty}f(s,y(s),\Delta y(s))\right).$$

and

$$(T_2x)(n) =$$

$$\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\gamma_n}\sum_{n=-\infty}^{+\infty}\gamma_n\sum_{t=-\infty}^{n-1}\frac{1}{\Psi^{-1}(q(t))}\Phi^{-1}\left(B_g(x)+\sum_{s=t}^{+\infty}g(s,x(s),\Delta x(s))\right)$$

$$+\sum_{t=-\infty}^{n-1}\frac{1}{\Psi^{-1}(q(t))}\Psi^{-1}\left(B_g(x)+\sum_{s=t}^{+\infty}g(s,x(s),\Delta x(s))\right),$$

where $B_g(x) \in \left[0, \epsilon_2 \sum\limits_{s=-\infty}^{+\infty} g(s,x(s),\Delta x(s))\right]$ satisfying

$$\Psi^{-1}(B_g(x)) =$$

$$\frac{\sum\limits_{n=-\infty}^{+\infty}\delta_n}{1-\sum\limits_{n=-\infty}^{+\infty}\gamma_n}\sum_{n=-\infty}^{+\infty}\gamma_n\sum_{t=-\infty}^{n-1}\frac{1}{\Psi^{-1}(q(t))}\Psi^{-1}\left(B_g(x)+\sum_{s=t}^{+\infty}g(s,x(s),\Delta x(s))\right)$$

$$+\sum_{n=-\infty}^{+\infty}\delta_n\sum_{t=-\infty}^{n-1}\frac{1}{\Psi^{-1}(q(t))}\Psi^{-1}\left(B_g(x)+\sum_{s=t}^{+\infty}g(s,x(s),\Delta x(s))\right).$$

LEMMA 8. *Suppose that (b′), (c′), (d′) and (e) hold. Then*

(i): *it holds that*

$$\begin{cases} \Delta[p(n)\Phi(\Delta(T_1y)(n))] + f(n,y(n),\Delta y(n)) = 0, & n \in \mathbf{Z}, \\ \Delta[q(n)\Psi(\Delta(T_2x)(n))] + g(n,x(n),\Delta x(n)) = 0, & n \in \mathbf{Z}, \\ \lim\limits_{n\to-\infty}(T_1y)(n) - \sum\limits_{n=-\infty}^{+\infty}\alpha_n(T_1y)(n) = 0, \\ \lim\limits_{n\to-\infty}(T_2x)(n) - \sum\limits_{n=-\infty}^{+\infty}\gamma_n(T_2x)(n) = 0, \\ \lim\limits_{n\to+\infty}\frac{(T_1y)(n)}{P_n} - \sum\limits_{n=-\infty}^{+\infty}\beta_n(T_1y)(n) = 0, \\ \lim\limits_{n\to+\infty}\frac{(T_2x)(n)}{Q_n} - \sum\limits_{n=-\infty}^{+\infty}\delta_n(T_2x)(n) = 0; \end{cases}$$

(ii): *$T(x,y) \in P$ for each $(x,y) \in P$;*

(iii): *(x,y) is a positive solution of BVP(5′) if and only if $(x,y) \in P$ is a fixed point of T in P;*

(iv): *$T : P \to P$ is completely continuous.*

Proof. For (i), (ii) and (iii), the proofs follow from Lemmas 4, 5, 6 and 7. We need to prove that (x, y) is a positive solution if (x, y) is a fixed point of T. In fact, by Lemmas 4-7, we know that both x and y are nonnegative. If there exist integers $M > m$ such that $x(n)^2 + y(n)^2 \equiv 0$ for $n \in [m, M]$, then we get $\Delta[p(n)\Phi(\Delta(T_1y)(n))] = \Delta[q(n)\Psi(\Delta(T_2x)(n))] = 0$ for $n \in [m, M-1]$. Hence $f(n,0,0)^2 + g(n,0,0)^2 = 0$ for all $n \in [m, M-1]$, a contradiction. Hence (x, y) is a positive solution of $(5')$.

(iv) It suffices to prove that T is continuous on P and T maps bounded subsets into relatively compact sets. We divide the proof into four steps:

Step 1: Prove that both $y \to A_f(y)$ and $x \to B_g(x)$ are continuous.

Let $(x_k, y_k) \in P$ with $y_k \to y_0$ and $x_k \to x_0$ as $k \to +\infty$. Then there exists positive number $r > 0$ such that

$$\sup_{n\in\mathbf{Z}} \frac{x_k(n)}{P_n},\ \sup_{n\in\mathbf{Z}} \frac{y_k(n)}{Q_n},\ \sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x_k(n)|,\ \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y_k(n)| \le r$$

for all $k = 0, 1, 2, \cdots$. Hence there exists a bilateral nonnegative sequence $\{\phi_r(n)\}$ with $\sum\limits_{n=-\infty}^{+\infty} \phi_r(n) + \infty$ satisfying

$$\begin{aligned} 0 &\le f(n, y_k(n), \Delta y_k(n)) \\ &= f\left(n, Q_n \tfrac{y_k(n)}{Q_n}, \tfrac{1}{\Psi^{-1}(q(n))}\Psi^{-1}(q(n))\Delta y_k(n)\right) \le \phi_r(n), \end{aligned}$$

and

$$\begin{aligned} 0 &\le g(n, x_k(n), \Delta x_k(n)) \\ &= g\left(n, P_n \tfrac{x_k(n)}{P_n}, \tfrac{1}{\Phi^{-1}(p(n))}\Phi^{-1}(p(n))\Delta x_k(n)\right) \le \phi_r(n). \end{aligned}$$

One sees that

$$0 \le A_f(y_k) \le \epsilon_1 \sum_{t=-\infty}^{+\infty} f(t, y_k(t), \Delta y_k(t)) \le \epsilon_1 \sum_{t=-\infty}^{+\infty} \phi_r(t).$$

We need to prove that $A_f(y_k) \to A_f(y_0)$ as $k \to +\infty$. If $A_f(y_k) \not\to A_f(y_0)$ as $k \to +\infty$, then there exist a sub-sequence such that

$A_f(y_{ki})^1 \to a_1 \neq A_f(y_0)$ as $i \to +\infty$. Then

$$\Phi^{-1}(A_f(y_{ki}^1)) =$$

$$\frac{\sum_{n=-\infty}^{+\infty}\beta_n}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A_f(y_{ki}^1)+\sum_{s=t}^{+\infty}f(s,y_{ki}^1(s),\Delta y_{ki}^1(s))\right)$$

$$+\sum_{n=-\infty}^{+\infty}\beta_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(A_f(y)+\sum_{s=t}^{+\infty}f(s,y_{ki}^1(s),\Delta y_{ki}^1(s))\right).$$

Let $i \to +\infty$, by using the generalized Leibegue dominated convergence theorem, we get that

$$\Phi^{-1}(a_1) =$$

$$\frac{\sum_{n=-\infty}^{+\infty}\beta_n}{1-\sum_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(a_1+\sum_{s=t}^{+\infty}f(s,y_0(s),\Delta y_0(s))\right)$$

$$+\sum_{n=-\infty}^{+\infty}\beta_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(a_1+\sum_{s=t}^{+\infty}f(s,y_0(s),\Delta y_0(s))\right).$$

From Lemma 2, we know that $a_1 = A_f(y_0)$, a contradiction. Hence $A_f(y_k) \to A_f(y_0)$ as $k \to +\infty$.

Step 2: Prove that both $T_1 : Y \to X$ and $T_2 : X \to Y$ are continuous.

Let $(x_k, y_k) \in P$ with $y_k \to y_0$ and $x_k \to x_0$ as $k \to +\infty$. We need to prove that $T_1y_k \to T_1y_0$ as $k \to +\infty$ and $T_2x_k \to T_2x_0$ as $k \to +\infty$. By Step 1, $A_f(y_k) \to A_f(y_0$ and $B_g(y_k) \to B_g(x_0)$ as $k \to +\infty$. This together with the continuous property of f, g implies that T is continuous at (x_0, y_0).

Step 3: For each bounded subset $\Omega \subset P$, prove that $T\Omega$ is bounded.

Since $\Omega \subset P$ is bounded, then there exists positive number $r > 0$ such that

(18)
$$\sup_{n\in\mathbf{Z}}\frac{x(n)}{P_n}, \sup_{n\in\mathbf{Z}}\frac{y(n)}{Q_n}, \sup_{n\in\mathbf{Z}}\Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}}\Psi^{-1}(q(n))|\Delta y(n)| \le r$$

for all $(x,y)\in\Omega$. Hence there exists a bilateral nonnegative sequence $\{\phi_r(n)\}$ with $\sum\limits_{n=-\infty}^{+\infty}\phi_r(n)+\infty$ satisfying

$$0\le f(n,y(n),\Delta y(n))\le \phi_r(n),\qquad 0\le g(n,x(n),\Delta x(n))\le \phi_r(n). \tag{19}$$

One sees that

$$0\le A_f(y)\le \epsilon_1\sum_{t=-\infty}^{+\infty} f(t,y(t),\Delta y(t))\le \epsilon_1\sum_{t=-\infty}^{+\infty}\phi_r(t)=:M_0,$$
$$0\le B_g(x)\le \epsilon_2\sum_{t=-\infty}^{+\infty} g(t,x(t),\Delta x(t))\le \epsilon_2\sum_{t=-\infty}^{+\infty}\phi_r(t)=M_0. \tag{20}$$

By the definitions of T_1 and T_2, we can prove that $T\Omega$ is bounded.

Step 4: For each bounded subset $\Omega\subset P$, prove that $T\Omega$ is relatively compact.

Since $\Omega\subset P$ is bounded, then there exists positive number $r>0$ such that (18) holds for all $(x,y)\in\Omega$. Hence there exists a bilateral nonnegative sequence $\{\phi_r(n)\}$ with $\sum\limits_{n=-\infty}^{+\infty}\phi_r(n)+\infty$ satisfying (19) and (20). Then

$$\left|\frac{(T_1y)(n)}{P_n}-\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{\Phi^{-1}\left(A_f(y)+\sum\limits_{s=t}^{+\infty}f(s,y(s),\Delta y(s))\right)}{\Phi^{-1}(p(t))}\right|$$

$$\le \frac{P_n-1}{P_n}\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{\Phi^{-1}\left((1+\epsilon_1)\sum\limits_{s=-\infty}^{+\infty}f(s,y(s),\Delta y(s))\right)}{\Phi^{-1}(p(t))}$$

$$+\frac{1}{P_n}\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left((1+\epsilon_1)\sum_{s=-\infty}^{+\infty}f(s,y(s),\Delta y(s))\right)$$

$$\leq \sum_{s=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(s))} \frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n} \sum_{n=-\infty}^{+\infty} \alpha_n \sum_{t=-\infty}^{n-1} \frac{\Phi^{-1}\left((1+\epsilon_1)\sum\limits_{s=-\infty}^{+\infty}\phi_r(s)\right)}{\Phi^{-1}(p(t))}$$

$$+\frac{1}{P_n}\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left((1+\epsilon_1)\sum_{s=-\infty}^{+\infty}\phi_r(s)\right)$$

$$\leq \Phi^{-1}\left((1+\epsilon_1)\sum_{s=-\infty}^{+\infty}\phi_r(s)\right)\left[\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}+1\right]\cdot$$

$\sum\limits_{s=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(s))} \to 0$ uniformly as $n \to -\infty$.

On the other hand, we have

$$\left|\frac{(T_1y)(n)}{P_n} - A_f(y)\right| \leq \frac{1}{P_n}\frac{\sum\limits_{n=-\infty}^{+\infty}\alpha_n\sum\limits_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left((1+\epsilon_1)\sum\limits_{s=-\infty}^{+\infty}\phi_r(s)\right)}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}$$

$$+\frac{1}{P_n}\sum_{t=-\infty}^{n-1}\frac{\Phi^{-1}\left(A_f(y)+\sum\limits_{s=t}^{+\infty}f(s,y(s),\Delta y(s))\right)}{\Phi^{-1}(p(t))} - \Phi^{-1}(A_f(y))$$

$$\leq \frac{1}{P_n}\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left((1+\epsilon_1)\sum_{s=-\infty}^{+\infty}\phi_r(s)\right)$$
$$+\frac{1}{P_n}\sum_{t=-\infty}^{n-1}\frac{\left|\Phi^{-1}\left(A_f(y)+\sum\limits_{s=t}^{+\infty}f(s,y(s),\Delta y(s))\right)-\Phi^{-1}(A_f(y))\right|}{\Phi^{-1}(p(t))} + \frac{\Phi^{-1}(A_f(y))}{P_n}$$
$$\leq \frac{1}{P_n}\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left((1+\epsilon_1)\sum_{s=-\infty}^{+\infty}\phi_r(s)\right)$$

$$+\frac{1}{P_n}\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\left|\Phi^{-1}\left(A_f(y)+\sum_{s=t}^{+\infty}f(s,y(s),\Delta y(s))\right)-\Phi^{-1}(A_f(y))\right|$$

$$+\frac{\Phi^{-1}\left((1+\epsilon_1)\sum\limits_{s=-\infty}^{+\infty}\phi_r(s)\right)}{P_n}.$$

Furthermore, we have

$$|\Phi^{-1}(p(n))\Delta(T_1y)(n) - \Phi^{-1}(A_f(y))| =$$

$$\left|\Phi^{-1}\left(A_f(y) + \sum_{s=n}^{+\infty} f(s, y(s), \Delta y(s))\right) - \Phi^{-1}(A_f(y))\right|$$

and

$$\left|\Phi^{-1}(p(n))\Delta(T_1y)(n) - \Phi^{-1}\left(A_f(y) + \sum_{t=-\infty}^{+\infty} f(t, y(t), \Delta y(t))\right)\right| =$$

$$\left|\Phi^{-1}\left(A_f(y) - \sum_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t))\right)\right.$$

$$\left.-\Phi^{-1}\left(A_f(y) - \sum_{t=-\infty}^{+\infty} f(t, y(t), \Delta y(t))\right)\right|.$$

For any $\epsilon > 0$, since Φ^{-1} is uniformly continuous on $[-2M_0, 2M_0]$, then there exists $\lambda > 0$ such that $|\Phi^{-1}(u_1) - \Phi^{-1}(u_2)| < \epsilon$ for all $_1, u_2 \in [-2M_0, 2M_0]$ with $|u_1 - u_2| < \lambda$.

From

$$\left|A_f(y) - \sum_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t)) - A_f(y)\right| \leq \sum_{t=-\infty}^{s-1} \phi_r(t),$$

$$\left|A_f(y) - \sum_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t)) - \left(A_f(y) - \sum_{t=-\infty}^{+\infty} f(t, y(t), \Delta y(t))\right)\right|$$

$$\leq \sum_{t=s}^{+\infty} \phi_r(t),$$

we know that there exists $N > 0$ such that

$$\left|A_f(y) - \sum_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t)) - A_f(y)\right| < \lambda \text{ uniformly as } s < -N,$$

$$\left|A_f(y) - \sum_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t)) - \left(A_f(y) - \sum_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t))\right)\right|$$

$$< \lambda \text{ uniformly as } s > N.$$

It follows that

$$|\Phi^{-1}(p(n))\Delta(T_1y)(n) - \Phi^{-1}(A_f(y))| < \epsilon \text{ uniformly as } s < -N,$$

$$\left|\Phi^{-1}(p(n))\Delta(T_1y)(n) - \Phi^{-1}\left(A_f(y) - \sum_{t=-\infty}^{+\infty} f(t, y(t), \Delta y(t))\right)\right|$$

$< \epsilon$ uniformly as $s > N$.

Then

$$\left|\frac{(T_1y)(n)}{P_n} - A_f(y)\right| \to 0 \text{ uniformly as } n \to +\infty.$$

So

$(T_1y)(n)$ is uniformly convergent as $n \to -\infty$,

$(T_1y)(n)$ is uniformly convergent as $n \to +\infty$,

$\Phi^{-1}(p(n))\Delta(T_1y)(n)$ is uniformly convergent as $n \to -\infty$,

$\Phi^{-1}(p(n))\Delta(T_1y)(n)$ is uniformly convergent as $n \to +\infty$.

Similarly, one has that

$(T_2x)(n)$ is uniformly convergent as $n \to -\infty$,

$(T_2x)(n)$ is uniformly convergent as $n \to +\infty$,

$\Psi^{-1}(q(n))\Delta(T_2x)(n)$ is uniformly convergent as $n \to -\infty$,

$\Psi^{-1}(q(n))\Delta(T_2x)(n)$ is uniformly convergent as $n \to +\infty$.

One knows that $T\Omega$ is relatively compact. Steps 1, 2, 3 and 4 imply that T is completely continuous.

□

Now, we establish existence of three positive solutions of BVP(5′) by using Lemma 1. Suppose that

$$\sup_{n\in\mathbf{Z}} \frac{1}{P_n} \sum_{t=n}^{+\infty} \alpha_t \sum_{s=n}^{t-1} \frac{1}{\Phi^{-1}(p(s))} < +\infty, \ \sup_{n\in\mathbf{Z}} \frac{1}{Q_n} \sum_{t=n}^{+\infty} \gamma_t \sum_{s=n}^{t-1} \frac{1}{\Psi^{-1}(q(s))} < +\infty.$$

Denote

$$M=\max\left\{\frac{1+\sum\limits_{t=-\infty}^{+\infty}\alpha_t+\sup\limits_{n\in\mathbf{Z}}\frac{1}{P_n}\sum\limits_{t=n}^{+\infty}\alpha_t\sum\limits_{s=n}^{t-1}\frac{1}{\Phi^{-1}(p(s))}}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n},\right.$$

$$\left.\frac{1+\sum\limits_{t=-\infty}^{+\infty}\gamma_t+\sup\limits_{n\in\mathbf{Z}}\frac{1}{Q_n}\sum\limits_{t=n}^{+\infty}\gamma_t\sum\limits_{s=n}^{t-1}\frac{1}{\Psi^{-1}(q(s))}}{1-\sum\limits_{n=-\infty}^{+\infty}\gamma_n}\right\}$$

and for positive numbers e_1, e_2, c and integers k_1, k_2, denote

$$W=\min\left\{\frac{2^{k-1}}{3\cdot 2^{k-1}-1}\Phi\left(\frac{e_2}{\frac{1}{P_k}\sum\limits_{t=-\infty}^{-k-1}\frac{1}{\Phi^{-1}(p(t))}}\right),\ \frac{2^{k-1}}{3\cdot 2^{k-1}-1}\Psi\left(\frac{e_2}{\frac{1}{Q_k}\sum\limits_{t=-\infty}^{-k-1}\frac{1}{\Psi^{-1}(q(t))}}\right)\right\};$$

$$Q=\max\left\{\frac{\Phi(c)}{3(\epsilon_1+1)},\ \frac{\Psi(c)}{3(\epsilon_2+1)},\ \frac{1}{3(\epsilon_1+1)}\Phi\left(\frac{c}{M}\right),\ \frac{1}{3(\epsilon_2+1)}\Psi\left(\frac{c}{M}\right)\right\},$$

$$E=\max\left\{\frac{1}{3(\epsilon_1+1)}\Phi\left(\frac{e_1}{M}\right),\ \frac{1}{3(\epsilon_2+1)}\Psi\left(\frac{e_1}{M}\right)\right\}.$$

THEOREM 1. *Let $k>0$ be an integer, $\mu=\min\{\mu_1,\mu_2\}$ with μ_1,μ_2 being defined by (15) and (17). Suppose that (b′), (c′), (d′) and (e) hold and there exist positive constants e_1, e_2, c such that*

$$c\geq\frac{e_2}{\mu}>e_2>e_1>0.$$

If $Q>W$ and

(A1): $\begin{array}{l} f\left(n, Q_n u, \frac{v}{\Psi^{-1}(q(n))}\right)\leq\frac{Q}{2^{|n|}} \\ g\left(n, P_n u, \frac{v}{\Phi^{-1}(p(n))}\right)\leq\frac{Q}{2^{|n|}}\end{array}$ *for all $n\in\mathbf{Z}, u\in[0,c], v\in[0,c]$;*

(A2): $\begin{array}{l} f\left(n, Q_n u, \frac{v}{\Psi^{-1}(q(n))}\right)\geq\frac{W}{2^{|n|}} \\ g\left(n, P_n u, \frac{v}{\Phi^{-1}(p(n))}\right)\geq\frac{W}{2^{|n|}}\end{array}$ *for all $n\in[-k,k], u\in[e_2,\frac{e_2}{\mu}], v\in$ $[0,c]$;*

(A3): $\begin{array}{l} f\left(n, Q_n u, \frac{v}{\Psi^{-1}(q(n))}\right)\leq\frac{E}{2^{|n|}} \\ g\left(n, P_n u, \frac{v}{\Phi^{-1}(p(n))}\right)\leq\frac{E}{2^{|n|}}\end{array}$ *for all $n\in\mathbf{Z}, u\in[0,e_1], v\in$ $[0,c]$.*

Then BVP(5′) has at least three positive solutions $(x_1, y_1), (x_2, y_2)$ *and* (x_3, y_3) *such that*

(21) $$\sup_{n\in\mathbf{Z}} x_1(n),\ \sup_{n\in Z} y_1(n) < e_1,\ \min_{n\in[-k,k]} x_2(n),\ \min_{n\in[-k,k]} y_2(n) > e_2,$$

and

(22) $$\text{either } \sup_{n\in\mathbf{Z}} x_3(n) \text{ or } \sup_{n\in\mathbf{Z}} y_3(n) > e_1,$$

(23) $$\text{either } \min_{n\in[-k,k]} x_3(n) \text{ or } \min_{n\in[-k,k]} y_3(n) < e_2.$$

Proof. Let E, P and T be defined in Section 2. We complete the proof of Theorem 1 by using Lemma 1. Define the following functionals by

(24)
$$\gamma(x,y) = \max\left\{\sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x(n)|,\ \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y(n)|\right\},$$

$$\beta(x,y) = \max\left\{\sup_{n\in\mathbf{Z}} \frac{x(n)}{P_n},\ \sup_{n\in\mathbf{Z}} \frac{y(n)}{Q_n}\right\}, (x,y)\in P,$$

$$\theta(x,y) = \max\left\{\sup_{n\in\mathbf{Z}} \frac{x(n)}{P_n},\ \sup_{n\in\mathbf{Z}} \frac{y(n)}{Q_n}\right\}, (x,y)\in P,$$

$$\alpha(x,y) = \min\left\{\min_{n\in[-k,k]} \frac{x(n)}{P_n},\ \min_{n\in[-k,k]} \frac{y(n)}{Q_n}\right\}, (x,y)\in P,$$

$$\varphi(x,y) = \min\left\{\min_{n\in[-k,k]} \frac{x(n)}{P_n},\ \min_{n\in[-k,k]} \frac{y(n)}{Q_n}\right\}, (x,y)\in P.$$

It is easy to see that α, ψ are two nonnegative continuous concave functionals on the cone P, γ, β, θ are three nonnegative continuous convex functionals on the cone P.

One sees $\alpha(x,y) \le \beta(x,y)$ for all $(x,y)\in P$. Lemmas in Section 2 imply that $(x,y) = (x(n), y(n))_{n=-\infty}^{+\infty}$ is a positive solution of BVP(5) if and only if (x,y) is a solution of the operator equation $(x,y) = T(x,y)$ and $T : P \to P$ is completely continuous.

By the definition of P, for $(x,y)\in P$, we have

$$\lim_{n\to-\infty} x(n) - \sum_{n=-\infty}^{+\infty} \alpha_n x(n) = 0,\ \lim_{n\to-\infty} y(n) - \sum_{n=-\infty}^{+\infty} \gamma_n y(n) = 0.$$

Then

$$0 \leq \frac{x(n)}{P_n} = \frac{1}{P_n}\frac{x(n)-\lim\limits_{n\to-\infty} x(n)+\lim\limits_{n\to-\infty} x(n)-x(n)\sum\limits_{n=-\infty}^{+\infty}\alpha_n}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}$$

$$= \frac{1}{P_n}\frac{\sum\limits_{t=-\infty}^{n-1}\Delta x(t)+\sum\limits_{t=-\infty}^{+\infty}\alpha_t x(t)-x(n)\sum\limits_{t=-\infty}^{+\infty}\alpha_t}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}$$

$$= \frac{1}{P_n}\frac{\sum\limits_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}(p(t))\Delta x(t)+\sum\limits_{t=-\infty}^{n-1}\alpha_t[x(t)-x(n)]+\sum\limits_{t=n}^{+\infty}\alpha_t[x(t)-x(n)]}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}$$

$$= \frac{1}{P_n}\frac{\sum\limits_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}(p(t))\Delta x(t)-\sum\limits_{t=-\infty}^{n-1}\alpha_t\sum\limits_{s=t}^{n-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}(p(s))\Delta x(s)}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}$$

$$+\frac{1}{P_n}\frac{\sum\limits_{t=n}^{+\infty}\alpha_t\sum\limits_{s=n}^{t-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}(p(s))\Delta x(s)}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}$$

$$\leq \frac{1+\sum\limits_{t=-\infty}^{+\infty}\alpha_t+\sup\limits_{n\in\mathbf{Z}}\frac{1}{P_n}\sum\limits_{t=n}^{+\infty}\alpha_t\sum\limits_{s=n}^{t-1}\frac{1}{\Phi^{-1}(p(s))}}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sup_{n\in\mathbf{Z}}\Phi^{-1}(p(n))|\Delta x(n)|$$

and

$$0 \leq \frac{y(n)}{Q_n} \leq \frac{1+\sum\limits_{t=-\infty}^{+\infty}\gamma_t+\sup\limits_{n\in Z}\frac{1}{Q_n}\sum\limits_{t=n}^{+\infty}\gamma_t\sum\limits_{s=n}^{t-1}\frac{1}{\Psi^{-1}(q(s))}}{1-\sum\limits_{n=-\infty}^{+\infty}\gamma_n}\sup_{n\in\mathbf{Z}}\Psi^{-1}(q(n))|\Delta y(n)|.$$

we have $||(x,y)|| \leq M\gamma(x,y)$ for all $(x,y) \in P$.

Corresponding to Lemma 1, choose

$$h = \mu e_1,\ \ d = e_1,\ \ a = e_2,\ \ b = \frac{e_2}{\mu},\ \ c = c.$$

Now, we prove that all conditions of Lemma 1 hold. One sees that $0 < d < a$. The remainder is divided into five steps.

Step 1: Prove that $T:\ \overline{P_c} \to \overline{P_c}$;

For $(x,y)\in\overline{P_c}$, we have $||(x,y)||\le c$. Then
$0\le\frac{x(n)}{P_n},\ \frac{y(n)}{Q_n}\le c, n\in\mathbf{Z}$,

$0\le\Phi^{-1}(p(n))|\Delta x(n)|,\Psi^{-1}(q(n))|\Delta y(n)|\le c$ for $n\in\mathbf{Z}$.

So (A1) implies that

$$f(n,y(n),\Delta y(n))=f\left(n,Q_n\frac{y(n)}{Q_n},\frac{\Psi^{-1}(q(n))\Delta y(n)}{\Psi^{-1}(q(n))}\right)\le\frac{Q}{2^{|n|}},\ n\in\mathbf{Z},$$

$$g(n,x(n),\Delta x(n))=g\left(n,P_n\frac{x(n)}{P_n},\frac{\Phi^{-1}(p(n))\Delta x(n)}{\Phi^{-1}(p(n))}\right)\le\frac{Q}{2^{|n|}},\ n\in\mathbf{Z}.$$

It follows from

$$0\le A_f(y)\le\epsilon_1\sum_{j=-\infty}^{+\infty}f(j,y(j),\Delta y(j)) \tag{25}$$

that

$$\Phi^{-1}(p(n))|\Delta(T_1y)(n)|=\left|\Phi^{-1}\left(A_f(y)+\sum_{j=n}^{+\infty}f(j,y(j),\Delta y(j))\right)\right|$$
$$\le\Phi^{-1}\left((\epsilon_1+1)\sum_{j=-\infty}^{+\infty}f(j,y(j),\Delta y(j))\right)$$
$$\le\Phi^{-1}\left((\epsilon_1+1)\sum_{j=-\infty}^{+\infty}\frac{Q}{2^{|j|}}\right)\le\Phi^{-1}\left(3(\epsilon_1+1)Q\right)\le c.$$

So

$$\sup_{n\in\mathbf{Z}}\Phi^{-1}(p(n))|\Delta(T_1y)(n)|\le c. \tag{26}$$

Then $T(x,y)\in P$ implies that

$$\frac{|(T_1y)(n)|}{P_n}\le M\sup_{n\in\mathbf{Z}}\phi^{-1}(p(n))|\Delta(T_1y)(n)|\le M\Phi^{-1}\left(6Q\right)\le c.$$

Hence

$$\sup_{n\in\mathbf{Z}}\frac{|(T_1y)(n)|}{P_n}\le c. \tag{27}$$

Similarly we can show that

$$\sup_{n\in\mathbf{Z}}\Psi^{-1}(q(n))|\Delta|(T_2x)(n))|\le c,\ \sup_{n\in\mathbf{Z}}\frac{|(T_2x)(n)|}{Q_n}\le c. \tag{28}$$

It follows from (26)-(28) that $||T(x,y)|| \leq c$. Then $T: \overline{P_c} \to \overline{P_c}$.

Step 2: Prove that

$$\{(x,y) \in P(\gamma,\theta,\alpha;a,b,c)|\alpha(x,y) > a\} =$$

$$\left\{(x,y) \in P\left(\gamma,\theta,\alpha;e_2,\tfrac{e_2}{\mu},c\right)|\alpha(x,y) > e_2\right\} \neq \emptyset$$

and $\alpha(T(x,y)) > e_2$ for every $(x,y) \in P\left(\gamma,\theta,\alpha;e_2,\frac{e_2}{\mu},c\right)$;

It is easy to show that $\{(x,y) \in P(\gamma,\theta,\alpha;a,b,c)|\alpha(x,y) > a\} \neq \emptyset$. For $(x,y) \in P(\gamma,\theta,\alpha;a,b,c)$, one has that

$$\alpha(x,y) = \min\left\{\min_{n\in[-k,k]} \frac{x(n)}{P_n}, \min_{n\in[-k,k]} \frac{y(n)}{Q_n}\right\} \geq e_2,$$

$$\theta(x) = \max\left\{\sup_{n\in\mathbf{Z}} \frac{x(n)}{P_n}, \sup_{n\in\mathbf{Z}} \frac{y(n)}{Q_n}\right\} \leq \frac{e_2}{\mu},$$

and

$$\gamma(x) = \max\left\{\sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y(n)|\right\} \leq c.$$

Then

$$e_2 \leq \frac{x(n)}{P_n}, \frac{y(n)}{Q_n} \leq \frac{e_2}{\mu},\ n \in [-k,k],$$

and

$$0 \leq \Phi^{-1}(p(n))|\Delta x(n)|, \Psi^{-1}(q(n))|\Delta y(n)| \leq c.$$

Thus (A2) implies that

$$f(n,y(n),\Delta y(n)), g(n,x(n),\Delta x(n)) \geq \frac{W}{2^{|n|}},\ \ n \in [-k,k].$$

So the definition of T_1 and $T_1 y \in X$ imply that

$$\min_{n\in[-k,k]} \frac{(T_1y)(n)}{P_n} \geq \frac{(T_1y)(-k)}{P_k}$$

$$\geq \frac{1}{P_k}\frac{1}{1-\sum\limits_{n=-\infty}^{+\infty}\alpha_n}\sum_{n=-\infty}^{+\infty}\alpha_n\sum_{t=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(\sum_{s=t}^{+\infty}f(s,y(s),\Delta y(s))\right)$$

$$+\frac{1}{P_k}\sum_{t=-\infty}^{-k-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(\sum_{s=t}^{+\infty}f(s,y(s),\Delta y(s))\right)$$

$$\geq \frac{1}{P_k}\sum_{t=-\infty}^{-k-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(\sum_{s=-k}^{k}f(s,y(s),\Delta y(s))\right)$$

$$\geq \frac{1}{P_k}\sum_{t=-\infty}^{-k-1}\frac{1}{\Phi^{-1}(p(t))}\Phi^{-1}\left(\sum_{s=-k}^{k}\frac{W}{2^{|s|}}\right).$$

So

$$\min_{n\in[-k,k]}\frac{(T_1y)(n)}{P_n} > e_2.$$

Similarly we can show that $\min\limits_{n\in[-k,k]}\frac{(T_2x)(n)}{Q_n} > e_2$. Then

$$\alpha(T(x,y)) = \min\left\{\min_{n\in[-k,k]}\frac{(T_1y)(n)}{P_n}, \min_{n\in[-k,k]}\frac{(T_2x)(n)}{Q_n}\right\} > e_2.$$

This completes Step 2.

Step 3: Prove that

$$\{(x,y)\in Q(\gamma,\theta,\varphi;h,d,c)|\beta(x,y)<d\} =$$

$$\{(x,y)\in Q\left(\gamma,\theta,\varphi;\mu e_1,e_1,c\right)|\beta(x,y)<e_1\} \neq \emptyset$$

and

$\beta(T(x,y)) < e_1$ for every $(x,y)\in Q(\gamma,\theta,\varphi;h,d,c) = Q\left(\gamma,\theta,\varphi;\mu e_1,e_1,c\right);$

Similarly to Step 2, we can see that $\{(x,y)\in Q(\gamma,\theta,\varphi;h,d,c)|$ $\beta(x,y)<d\}\neq\emptyset$.

For $(x,y)\in Q(\gamma,\theta,\varphi;h,d,c)$, one has that

$$\varphi(x,y) = \min\left\{\min_{n\in[-k,k]}\frac{x(n)}{P_n}, \min_{n\in[-k,k]}\frac{y(n)}{Q_n}\right\} \geq \mu e_1$$

$$\theta(x,y) = \max\left\{\sup_{n\in\mathbf{Z}}\frac{x(n)}{P_n}, \sup_{n\in\mathbf{Z}}\frac{y(n)}{Q_n}\right\} \le d = e_1,$$

and

$$\gamma(x,y) = \max\left\{\sup_{n\in\mathbf{Z}}\Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}}\Psi^{-1}(q(n))|\Delta y(n)|\right\} \le c.$$

Hence we get that

$$0 \le \frac{x(n)}{P_n}, \frac{y(n)}{Q_n} \le e_1, 0 \le \Phi^{-1}(p(n))|\Delta x(n)|, \Psi^{-1}(q(n))|\Delta y(n)| \le c,\ n\in\mathbf{Z}.$$

Then (A3) implies that

$$f(n, y(n), \Delta y(n)), g(n, x(n), \Delta x(n)) \le \frac{E}{2^{|n|}},\quad n\in\mathbf{Z}.$$

So similarly to Step 1, it follows that

$$\begin{aligned}&\beta(T(x,y))\\ &\le M\max\left\{\sup_{n\in\mathbf{Z}}\Phi^{-1}(p(n))|\Delta(T_1y)(n)|, \sup_{n\in\mathbf{Z}}\Psi^{-1}(q(n))|\Delta(T_2x)(n)|\right\}\\ &\le M\max\left\{\Phi^{-1}(3(\epsilon_1+1)E), \Psi^{-1}(3(\epsilon_2+1)E)\right\} \le e_1 = d.\end{aligned}$$

This completes Step 3.

Step 4: Prove that $\alpha(T(x,y)) > a$ for $(x,y)\in P(\gamma,\alpha;a,c)$ with $\theta(T(x,y)) > b$;

For $(x,y)\in P(\gamma,\alpha;a,c) = P(\gamma,\alpha;e_2,c)$ with $\theta(T(x,y)) = \beta(T(x,y)) > b = \frac{e_2}{\mu}$, we have that

$$\begin{aligned}\alpha(x,y) &= \min\left\{\min_{n\in[-k,k]}\frac{x(n)}{P_n}, \min_{n\in[-k,k]}\frac{y(n)}{Q_n}\right\} \ge e_2,\\ \gamma(x,y) &= \max\left\{\sup_{n\in Z}\Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}}\Psi^{-1}(q(n))|\Delta y(n)|\right\} \le c,\\ &\quad \sup_{n\in\mathbf{Z}}(T(x,y))(n) > \frac{e_2}{\mu}.\end{aligned}$$

Then

$$\alpha(T(x,y)) = \min_{n\in[-k,k]}(T(x,y))(n) \ge \mu\beta(T(x,y)) > e_2 = a.$$

This completes Step 4.

Step 5: Prove that $\beta(T(x,y)) < d$ for each $(x,y) \in Q(\gamma,\beta;d,c)$ with $\varphi(Tx) < h$.

For $(x,y) \in Q(\gamma,\beta;d,c)$ with $\varphi(Tx) < h$, we have

$$\begin{aligned}
\gamma(x,y) &= \max\left\{\sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y(n)|\right\} \le c,\\
\beta(x,y) &= \max\left\{\sup_{n\in\mathbf{Z}} \frac{x(n)}{P_n}, \sup_{n\in\mathbf{Z}} \frac{y(n)}{Q_n}\right\} \le d = e_1,\\
\varphi(T(x,y)) &= \min\left\{\min_{n\in[-k,k]} \frac{(T_1y)(n)}{P_n}, \min_{n\in[-k,k]} \frac{(T_2x)(n)}{Q_n}\right\} < h = \mu e_1.
\end{aligned}$$

Then

$$\beta(T(x,y)) \le \frac{1}{\mu}\varphi(T(x,y)) < e_1 = d.$$

This completes the Step 5.

Then Lemma 1 implies that T has at least three fixed points (x_1,y_1), (x_2,y_2) and (x_3,y_3) such that

$$\beta(x_1,y_1) < e_1,\ \alpha(x_2,y_2) > e_2,\ \beta(x_3,y_3) > e_1,\ \alpha(x_3,y_3) < e_2.$$

Hence BVP(5′) has three positive solutions $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) satisfying (21)-(23). The proof is complete. □

4. Existence of positive solutions of BVP(5)

In this section, we establish existence result for three positive solutions of BVP(5).

Choose

$$X = \left\{ \{x(n) : n \in \mathbf{Z}\} : \begin{array}{c} x(n) \in \mathbb{R}, n \in \mathbf{Z}, \\ \text{there exist the limits} \\ \lim_{n\to+\infty} x(n),\ \lim_{n\to-\infty} x(n), \\ \lim_{n\to+\infty} \Phi^{-1}(p(n))\Delta x(n), \\ \lim_{n\to-\infty} \Phi^{-1}(p(n))\Delta x(n) \end{array} \right\}.$$

Define the norm

$$||x||_X = ||x|| = \max\left\{\sup_{n\in\mathbf{Z}} |x(n)|,\ \sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x(n)|\right\}, x \in X.$$

It is easy to see that X is a real Banach space.

Choose

$$Y=\left\{\{y(n):n\in\mathbf{Z}\}:\begin{array}{c}y(n)\in\mathbb{R},n\in\mathbf{Z},\\ \text{there exist the limits}\\ \lim\limits_{n\to+\infty}y(n),\ \lim\limits_{n\to-\infty}y(n),\\ \lim\limits_{n\to+\infty}\Psi^{-1}(q(n))\Delta y(n),\\ \lim\limits_{n\to-\infty}\Psi^{-1}(q(n))\Delta y(n)\end{array}\right\}.$$

Define the norm

$$||y||_Y=||y||=\max\left\{\sup_{n\in\mathbf{Z}}|y(n)|,\ \sup_{n\in\mathbf{Z}}\Psi^{-1}(q(n))|\Delta y(n)|\right\},y\in Y.$$

It is easy to see that Y is a real Banach space.

Let $E=X\times Y$ be defined by $E=\{(x,y):x\in X,y\in Y\}$ with the norm $||(x,y)||=\max\{||x||,||y||\}$. Then E is a Banach space.

LEMMA 9. *Suppose that (b), (c) and (e) hold and $h(n)\not\equiv 0(n\in\mathbf{Z})$ be a nonnegative sequence with $\sum\limits_{n=-\infty}^{+\infty}h(n)$ converging. Then there exists a unique number $A_h\in\left[0,\sum\limits_{n=-\infty}^{+\infty}h(n)\right]$ such that*

$$\begin{aligned}&\frac{1-\sum_{i=-\infty}^{+\infty}\alpha_i}{1-\sum_{i=-\infty}^{+\infty}\beta_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right)\\ &+\sum_{s=-\infty}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right)\\ &=\sum_{i=-\infty}^{+\infty}\beta_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right).\end{aligned}\tag{29}$$

Proof. Let

$$G(w)=\frac{1-\sum\limits_{i=-\infty}^{+\infty}\alpha_i}{1-\sum\limits_{i=-\infty}^{+\infty}\beta_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(w-\sum_{t=-\infty}^{s-1}h(t)\right)$$

$$+\sum_{s=-\infty}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(w-\sum_{t=-\infty}^{s-1}h(t)\right)$$

$$-\sum_{i=-\infty}^{+\infty}\beta_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(w-\sum_{t=-\infty}^{s-1}h(t)\right).$$

Then

$$G(w)=\frac{1-\sum\limits_{i=-\infty}^{+\infty}\alpha_i}{1-\sum\limits_{i=-\infty}^{+\infty}\beta_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(w-\sum_{t=-\infty}^{s-1}h(t)\right)$$

$$+\left(1-\sum_{i=-\infty}^{+\infty}\beta_i\right)\sum_{s=-\infty}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(w-\sum_{t=-\infty}^{s-1}h(t)\right)$$

$$+\sum_{i=-\infty}^{+\infty}\beta_i\sum_{s=i}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(w-\sum_{t=-\infty}^{s-1}h(t)\right).$$

It is easy to see that G is increasing on R. Since $G(0)\le 0$ and $G\left(\sum\limits_{n=-\infty}^{+\infty}h(n)\right)\ge 0$, then there exists a unique $A_h\in\left[0,\sum\limits_{n=-\infty}^{+\infty}h(n)\right]$ such that (29) holds. The proof is completed. □

Let $\{h(n)\}$ be a nonnegative sequence with $\sum\limits_{n=-\infty}^{+\infty}h(n)$ converging and satisfies that for each $n_0\in\mathbf{Z}$, $h(n)\not\equiv 0$ for $n\le n_0$, for each $n_1\in\mathbf{Z}$, $h(n)\not\equiv 0$ for $n\ge n_1$.

Consider the following BVP

$$\text{(30)}\qquad\begin{cases}\Delta[p(n)\Phi(\Delta x(n))]+h(n)=0,\quad n\in\mathbf{Z},\\ \lim\limits_{n\to-\infty}x(n)-\sum\limits_{n=-\infty}^{+\infty}\alpha_n x(n)=0,\\ \lim\limits_{n\to+\infty}x(n)-\sum\limits_{n=-\infty}^{+\infty}\beta_n x(n)=0,\end{cases}$$

Lemma 10. *Suppose that (b), (c) and (e) hold. Then x is a nonnegative solution of BVP(28) if and only if*

$$x(n) = \frac{1}{1-\sum_{i=-\infty}^{+\infty}\alpha_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right) \tag{31}$$
$$+\sum_{s=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right)$$

where $A_h \in \left[0, \sum_{s=-\infty}^{+\infty}h(s)\right]$ *satisfying (29).*

Proof. Suppose that x is a solution of (30). Form (30), we know that there exist $A, B \in R$ such that

$$p(n)\Phi(\Delta x(n)) = A - \sum_{t=-\infty}^{n-1}h(s),$$

and

$$x(n) = B + \sum_{s=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A-\sum_{t=-\infty}^{s-1}h(t)\right), n \in Z.$$

By the first boundary condition in (30), we get

$$B = B\sum_{i=-\infty}^{+\infty}\alpha_i + \sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A-\sum_{t=-\infty}^{s-1}h(t)\right).$$

It follows that

$$B = \frac{1}{1-\sum_{i=-\infty}^{+\infty}\alpha_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A-\sum_{t=-\infty}^{s-1}h(t)\right).$$

The second boundary condition in (30) implies that

$$\frac{1-\sum_{i=-\infty}^{+\infty}\alpha_i}{1-\sum_{i=-\infty}^{+\infty}\beta_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right)$$
$$+\sum_{s=-\infty}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right)$$
$$=\sum_{i=-\infty}^{+\infty}\beta_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right).$$

It follows from Lemma 9 that there exists a unique $A_h \in \left[0, \sum\limits_{s=-\infty}^{+\infty} h(s)\right]$ such that (29) holds. Hence $A = A_h$. So

$$B = \frac{1}{1-\sum_{i=-\infty}^{+\infty}\alpha_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}\left(p(s)\right)}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right).$$

Then $x(n)$ satisfies (31) with A_h satisfying (29). Now we prove that $x \in X$ and x is nonnegative.

In fact, we can prove that

$$\lim_{n\to+\infty} x(n) = \frac{1}{1-\sum_{i=-\infty}^{+\infty}\alpha_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right)$$

$$+\sum_{s=-\infty}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right),$$

$$\lim_{n\to-\infty} x(n) = \frac{1}{1-\sum_{i=-\infty}^{+\infty}\alpha_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_h-\sum_{t=-\infty}^{s-1}h(t)\right),$$

$$\lim_{n\to-\infty}\Phi^{-1}(p(n))\Delta x(n) = \Phi^{-1}(A_h),$$

$$\lim_{n\to+\infty}\Phi^{-1}(p(n))\Delta x(n) = \Phi^{-1}\left(A_h-\sum_{n=-\infty}^{+\infty}h(n)\right).$$

Hence $x \in X$.

Since $\Delta[p(n)\Phi(\Delta x(n))] = -h(n) \le 0$ for all $n \in \mathbf{Z}$ and $\sum\limits_{n\in\mathbf{Z}} h(n)$ converges, we get that $p(n)\Phi(\Delta x(n))$ is decreasing and there exists the limit $\lim\limits_{n\to\infty} p(n)\Phi(\Delta x(n))$. Then $\Phi^{-1}(p(n))\Delta x(n)$ is decreasing. We will prove that $\lim\limits_{n\to+\infty}\Phi^{-1}(p(n))\Delta x(n) < 0$ and $\lim\limits_{n\to-\infty}\Phi^{-1}(p(n))\Delta x(n) > 0$.

In fact, if $\lim\limits_{n\to+\infty}\Phi^{-1}(p(n))\Delta x(n) \ge 0$, then $\Phi^{-1}(p(n))\Delta x(n) \ge 0$. So $\Delta x(n) \ge 0$ for all $n \in \mathbf{Z}$. Then x is increasing. So we get that

$$\lim_{n\to-\infty} x(n) = \sum_{n=-\infty}^{\infty}\alpha_n x(n) \ge \sum_{n=-\infty}^{\infty}\alpha_n \lim_{n\to-\infty} x(n).$$

It follows that

$$\left(1-\sum_{n=-\infty}^{\infty}\alpha_n\right)\lim_{n\to-\infty} x(n) \ge 0.$$

Hence $\lim_{n\to-\infty} x(n) \geq 0$. It follows that $x(n) \geq 0$ for all $n \in \mathbf{Z}$.

On the other hand, we have

$$\lim_{n\to+\infty} x(n) = \sum_{n=-\infty}^{+\infty} \beta_n x(n) \leq \sum_{n=-\infty}^{+\infty} \beta_n \lim_{n\to+\infty} x(n).$$

We get similarly that $x(n) \leq 0$ for all $n \in \mathbf{Z}$, a contradiction. So

$$\lim_{n\to+\infty} \Phi^{-1}(p(n))\Delta x(n) < 0.$$

Similarly we can show that $\lim_{n\to-\infty} \Phi^{-1}(p(n))\Delta x(n) > 0$.

Then there exists $n_2 \in \mathbf{Z}$ such that

$\Phi^{-1}(p(n))\Delta x(n) \geq 0$ for all $n \leq n_2$ and $\Phi^{-1}(p(n))\Delta x(n) < 0$ for all $n \geq n_2+1$.

Then (9) implies that

$$p(n)\Phi(\Delta x(n)) = \begin{cases} \Phi^{-1}(p(n_2))\Delta x(n_2) + \sum_{s=n}^{n_2-1} h(s), \ n \leq n_2, \\ \\ \Phi^{-1}(p(n_2+1))\Delta x(n_2+1) - \sum_{s=n_2+1}^{n-1} h(s), \ n \geq n_2+1. \end{cases}$$

So

$$x(n) = \begin{cases} \lim_{n\to-\infty} x(n) + \\ \sum_{s=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(s))} \Phi^{-1}\left(\Phi^{-1}(p(n_2))\Delta x(n_2) + \sum_{t=s}^{n_2-1} h(t)\right), n \leq n_2+1, \\ \\ \lim_{n\to+\infty} x(n) - \\ \sum_{s=n}^{+\infty} \frac{1}{\Phi^{-1}(p(s))} \Phi^{-1}\left(\Phi^{-1}(p(n_2+1))\Delta x(n_2+1) - \sum_{t=n_2+1}^{s-1} h(t)\right), \\ \qquad n \geq n_2+1. \end{cases}$$

It follows that

$$x(n) \geq \min\left\{\lim_{n\to-\infty} x(n), \lim_{n\to+\infty} x(n)\right\}.$$

Thus (9) implies that

$$\lim_{n\to-\infty} x(n) \geq \sum_{n=-\infty}^{+\infty} \alpha_n \min\left\{\lim_{n\to-\infty} x(n), \lim_{n\to+\infty} x(n)\right\},$$

$$\lim_{n\to+\infty} x(n) \geq \sum_{n=-\infty}^{+\infty} \beta_n \min\left\{\lim_{n\to-\infty} x(n), \lim_{n\to+\infty} x(n)\right\}.$$

It is easy to see that

$$\lim_{n\to-\infty} x(n) \geq 0, \quad \lim_{n\to+\infty} x(n) \geq 0.$$

Then $x(n) \geq 0$ for all $n \in \mathbf{Z}$. Now we prove that $x(n) > 0$ for all $n \in \mathbf{Z}$. In fact, if there exists $x(n_3) \leq 0$, then either $n_3 \leq n_2$ or $n_3 \geq n_2 + 1$. If $n_3 \leq n_2$, then we get from $\Delta x(n) \geq 0$ for all $n \leq n_2$ that $x(n) \equiv 0$ for all $n \leq n_2$. Hence $h(n) \equiv 0$ for all $n \leq n_2$, a contradiction. Similarly we get a contradiction for $n_3 \geq n_2 + 1$. Hence $x(n) > 0$ for all $n \in \mathbf{Z}$.

On the other hand, if x satisfies (31) with A_h satisfying (29), then we can prove that x is a nonnegative solution of (30), we omit the details. The proof is completed. $\square$

Consider the following BVP

$$\begin{cases} \Delta[q(n)\Psi(\Delta y(n))] + h(n) = 0, \quad n \in \mathbf{Z}, \\ \lim_{n\to-\infty} y(n) - \sum_{n=-\infty}^{+\infty} \gamma_n y(n) = 0, \\ \lim_{n\to+\infty} y(n) - \sum_{n=-\infty}^{+\infty} \delta_n y(n) = 0, \end{cases} \tag{32}$$

Similarly we can prove the following three lemmas:

Lemma 11. *Suppose that (b), (c) and (e) hold and $h(n) \not\equiv 0 (n \in \mathbf{Z})$ be a nonnegative sequence with $\sum_{n=-\infty}^{+\infty} h(n)$ converging. Then there exists* *a unique number* $B_h \in \left[0, \sum\limits_{n=-\infty}^{+\infty} h(n)\right]$ *such that*

$$\begin{aligned}&\tfrac{1-\sum_{i=-\infty}^{+\infty}\gamma_i}{1-\sum_{i=-\infty}^{+\infty}\delta_i}\sum_{i=-\infty}^{+\infty}\gamma_i\sum_{s=-\infty}^{i-1}\tfrac{1}{\Psi^{-1}(q(s))}\Psi^{-1}\left(B_h-\sum_{t=-\infty}^{s-1}h(t)\right)\\ &+\sum_{s=-\infty}^{+\infty}\tfrac{1}{\Psi^{-1}(q(s))}\Psi^{-1}\left(B_h-\sum_{t=-\infty}^{s-1}h(t)\right)\\ &=\sum_{i=-\infty}^{+\infty}\delta_i\sum_{s=-\infty}^{i-1}\tfrac{1}{\Psi^{-1}(q(s))}\Psi^{-1}\left(B_h-\sum_{t=-\infty}^{s-1}h(t)\right).\end{aligned}\tag{33}$$

Lemma 12. *Suppose that (b), (c) and (e) hold. Then y is a positive solution of BVP(32) if and only if*

$$\begin{aligned}y(n)=&\tfrac{1}{1-\sum\limits_{i=-\infty}^{+\infty}\gamma_i}\sum_{i=-\infty}^{+\infty}\gamma_i\sum_{s=-\infty}^{i-1}\tfrac{1}{\Psi^{-1}(q(s))}\Psi^{-1}\left(B_h-\sum_{t=-\infty}^{s-1}h(t)\right)\\ &+\sum_{s=-\infty}^{n-1}\tfrac{1}{\Psi^{-1}(q(s))}\Psi^{-1}\left(B_h-\sum_{t=-\infty}^{s-1}h(t)\right)\end{aligned}\tag{34}$$

where $B_h \in \left[0, \sum\limits_{s=-\infty}^{+\infty} h(s)\right]$ *satisfying (33).*

Lemma 13. *Choose integers $k_1, k_2 \in \mathbf{Z}$ with $k_1 + 3 < k_2$. Suppose that (b), (c) and (e) hold, $h(n) \geq 0$ for all $n \in \mathbf{Z}$. Suppose x is a solution of BVP(28). Then*

$$\min_{n\in[k_1,k_2]} x(n) \geq \mu_1 \sup_{n\in\mathbf{Z}} x(n),\tag{35}$$

where

$$\mu_1 = \min\left\{\frac{P_{k_1}-P_{k_1-1}}{P_{+\infty}-P_{k_1-1}}, \frac{P_{k_2+1}-P_{k_2}}{P_{k_2+1}}\right\}.\tag{36}$$

Proof. Since $\Delta[p(n)\Phi(\Delta x(n))] = -h(n) \leq 0$ for all $n \in \mathbf{Z}$, we see that $p(n)\Phi(\Delta x(n))$ is decreasing. Then $\Phi^{-1}(p(n))\Delta x(n)$ is decreasing.

It follow from Lemma 10 that $x(n) \geq 0$ for all $n \in \mathbf{Z}$. For $n_1, n, n_2 \in \mathbf{Z}$ with $n_1 < n < n_2$, Since $\Phi^{-1}(p(n))\Delta x(n)$ is decreasing, we get

$$\Phi^{-1}(p(s))\Delta x(s) \leq \Phi^{-1}(p(k))\Delta x(k)$$

for all $s \geq k$. So there there is λ such that

$$\Phi^{-1}(p(s))\Delta x(s) \leq \lambda \leq \Phi^{-1}(p(k))\Delta x(k),\ s \geq n > k.$$

Then we get (16) similarly to the proof of Lemma 6. We will use (16) to complete the proof of (35). We consider three cases:

Case 1: there is $n_0 \in \mathbf{Z}$ such that $\sup_{n\in\mathbf{Z}} x(n) = x(n_0)$.

If $n_0 = k_1$, we get by using (13) that

$$\begin{aligned}
&\min_{n\in[k_1,k_2]} x(n) = x(k_2)\\
\geq\ & \frac{P_{k_2+1}-P_{k_2}}{P_{k_2+1}-P_{k_1}}x(k_1) + \frac{P_{k_2}-P_{k_1}}{P_{k_2+1}-P_{k_1}}x(k_2+1)\\
\geq\ & \frac{P_{k_2+1}-P_{k_2}}{P_{k_2+1}-P_{k_1}}x(n_0)\\
\geq\ & \mu_1 \sup_{n\in\mathbf{Z}} x(n).
\end{aligned}$$

If $n_0 = k_2$, we get by using (13) that

$$\begin{aligned}
&\min_{n\in[k_1,k_2]} x(n) = x(k_1)\\
\geq\ & \frac{P_{k_2}-P_{k_1}}{P_{k_2}-P_{k_1-1}}x(k_1-1) + \frac{P_{k_1}-P_{k_1-1}}{P_{k_2}-P_{k_1-1}}x(k_2)\\
\geq\ & \frac{P_{k_1}-P_{k_1-1}}{P_{k_2}-P_{k_1-1}}x(n_0)\\
\geq\ & \mu_1 \sup_{n\in Z} x(n).
\end{aligned}$$

If $n_0 > k_2$, for $n \in [k_1-1, n_0]$, by using (11) we have

$$x(n) \geq \tfrac{P_{n_0}-P_n}{P_{n_0}-P_{k_1-1}}x(k_1-1) + \tfrac{P_n-P_{k_1-1}}{P_{n_0}-P_{k_1-1}}x(n_0) \geq \tfrac{P_n-P_{k_1-1}}{P_{n_0}-P_{k_1-1}}x(n_0).$$

It follows for $n \in [k_1, k_2]$ that

$$x(n) \geq \frac{P_{k_1}-P_{k_1-1}}{P_{+\infty}-P_{k_1-1}}x(n_0).$$

Then

$$\min_{n\in[k_1,k_2]} x(n) \geq \frac{P_{k_1}-P_{k_1-1}}{P_{+\infty}-P_{k_1-1}}x(n_0) \geq \mu_1 \sup_{n\in\mathbf{Z}} x(n).$$

If $n_0 < k_1$, for $n \in [n_0, k_2+1]$, by using (11) we have

$$x(n) \geq \tfrac{P_{k_2+1}-P_n}{P_{k_2+1}-P_{n_0}}x(n_0) + \tfrac{P_n-P_{n_0}}{P_{k_2+1}-P_{n_0}}x(k_2+1) \geq \tfrac{P_{k_2+1}-P_n}{P_{k_2+1}-P_{n_0}}x(n_0).$$

It follows for $n \in [k_1, k_2]$ that

$$x(n) \geq \frac{P_{k_2+1} - P_{k_2}}{P_{k_2+1}} x(n_0).$$

Then

$$\min_{n \in [k_1, k_2]} x(n) \geq \frac{P_{k_2+1} - P_{k_2}}{P_{k_2+1}} x(n_0) \geq \mu_1 \sup_{n \in \mathbf{Z}} x(n).$$

If $k_1 < n_0 < k_2$, for $n \in [k_1 - 1, n_0]$, by using (11) we have

$$x(n) \geq \tfrac{P_{n_0} - P_n}{P_{n_0} - P_{k_1-1}} x(k_1 - 1) + \tfrac{P_n - P_{k_1-1}}{P_{n_0} - P_{k_1-1}} x(n_0) \geq \tfrac{P_n - P_{k_1-1}}{P_{n_0} - P_{k_1-1}} x(n_0).$$

It follows for $n \in [k_1, n_0]$ that

$$x(n) \geq \frac{P_{k_1} - P_{k_1-1}}{P_{+\infty} - P_{k_1-1}} x(n_0).$$

For $n \in [n_0, k_2 + 1]$, by using (11) we have

$$x(n) \geq \tfrac{P_{k_2+1} - P_n}{P_{k_2+1} - P_{n_0}} x(n_0) + \tfrac{P_n - P_{n_0}}{P_{k_2+1} - P_{n_0}} x(k_2 + 1) \geq \tfrac{P_{k_2+1} - P_n}{P_{k_2+1} - P_{n_0}} x(n_0).$$

Then for $n \in [n_0, k_2]$ we have

$$x(n) \geq \frac{P_{k_2+1} - P_{k_2}}{P_{k_2+1}} x(n_0).$$

It follows for $n \in [k_1, k_2]$ that

$$x(n) \geq \min \left\{ \frac{P_{k_1} - P_{k_1-1}}{P_{+\infty} - P_{k_1-1}}, \frac{P_{k_2+1} - P_{k_2}}{P_{k_2+1}} \right\} x(n_0).$$

Then

$$\min_{n \in [k_1, k_2]} x(n) \geq \min \left\{ \frac{P_{k_1} - P_{k_1-1}}{P_{+\infty} - P_{k_1-1}}, \frac{P_{k_2+1} - P_{k_2}}{P_{k_2+1}} \right\} x(n_0) \geq \mu_1 \sup_{n \in \mathbf{Z}} x(n).$$

Case 2: $\sup_{n \in \mathbf{Z}} x(n) = \lim_{n \to +\infty} x(n)$.

Choose $n' > k_2$, similarly we can prove that

$$\min_{n \in [k_1, k_2]} x(n) \geq \mu_1 x(n').$$

Let $n' \to +\infty$, one sees

$$\min_{n \in [k_1, k_2]} x(n) \geq \mu_1 \sup_{n \in \mathbf{Z}} x(n).$$

Case 3: $\sup_{n\in\mathbf{Z}} x(n) = \lim_{n\to-\infty} x(n)$.

Choose $n' < k_1$, similarly we can prove that

$$\min_{n\in[k_1,k_2]} x(n) \geq \mu_1 x(n').$$

Let $n' \to -\infty$, one sees

$$\min_{n\in[k_1,k_2]} x(n) \geq \mu_1 \sup_{n\in\mathbf{Z}} x(n).$$

From Cases 1, 2 and 3, we get (11). The proof is complete.

□

LEMMA 14. *Choose integers* $k_1, k_2 \in \mathbf{Z}$ *with* $k_1 + 3 < k_2$. *Suppose that (b), (c) and (e) hold,* $h(n) \geq 0$ *for all* $n \in Z$. *Suppose* x *is a solution of BVP(30). Then*

$$\min_{n\in[k_1,k_2]} y(n) \geq \mu_2 \sup_{n\in\mathbf{Z}} y(n), \tag{37}$$

where

$$\mu_2 = \min\left\{\frac{Q_{k_1} - Q_{k_1-1}}{Q_{+\infty} - Q_{k_1-1}}, \frac{Q_{k_2+1} - Q_{k_2}}{Q_{k_2+1}}\right\}. \tag{38}$$

Let $\mu = \min\{\mu_1, \mu_2\}$. Define the cone P in $X \times Y = E$ by

$$P = \left\{ (x,y) \in E : \begin{array}{c} x(n), y(n) \geq 0, n \in \mathbf{Z}, \\ \lim\limits_{n\to-\infty} x(n) - \sum\limits_{n=-\infty}^{+\infty} \alpha_n x(n) = 0, \\ \lim\limits_{n\to-\infty} y(n) - \sum\limits_{n=-\infty}^{+\infty} \gamma_n y(n) = 0, \\ \lim\limits_{n\to+\infty} x(n) - \sum\limits_{n=-\infty}^{+\infty} \beta_n \Delta x(n) = 0, \\ \lim\limits_{n\to+\infty} y(n) - \sum\limits_{n=-\infty}^{+\infty} \delta_n \Delta y(n) = 0, \\ \min\limits_{n\in[k_1,k_2]} x(n) \geq \mu \sup\limits_{n\in\mathbf{Z}} x(n), \\ \min\limits_{n\in[k_1,k_2]} y(n) \geq \mu \sup\limits_{n\in\mathbf{Z}} y(n) \end{array} \right\}.$$

For $(x,y) \in P$, define $(T(x,y))(n) = ((T_1y)(n),(T_2x)(n))$ by

$$(T_1y)(n) = \frac{\sum\limits_{i=-\infty}^{+\infty}\alpha_i \sum\limits_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y) - \sum\limits_{t=-\infty}^{s-1} f(t,y(t),\Delta y(t))\right)}{1-\sum\limits_{i=-\infty}^{+\infty}\alpha_i}$$

$$+ \sum_{s=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y) - \sum_{t=-\infty}^{s-1} f(t,y(t),\Delta y(t))\right), n \in \mathbf{Z},$$

where $A_f(y) \in \left[0, \sum\limits_{s=-\infty}^{+\infty} f(s,y(s),\Delta y(s))\right]$ satisfying

$$\frac{1-\sum\limits_{i=-\infty}^{+\infty}\alpha_i}{1-\sum\limits_{i=-\infty}^{+\infty}\beta_i}\sum_{i=-\infty}^{+\infty}\alpha_i \sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y) - \sum_{t=-\infty}^{s-1} f(t,y(t),\Delta y(t))\right)$$

$$+ \sum_{s=-\infty}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y) - \sum_{t=-\infty}^{s-1} f(t,y(t),\Delta y(t))\right)$$

$$= \sum_{i=-\infty}^{+\infty}\beta_i \sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y) - \sum_{t=-\infty}^{s-1} f(t,y(t),\Delta y(t))\right)$$

and

$$(T_2x)(n) = \frac{\sum\limits_{i=-\infty}^{+\infty}\gamma_i \sum\limits_{s=-\infty}^{i-1}\frac{1}{\Psi^{-1}(q(s))}\Psi^{-1}\left(B_g(x) - \sum\limits_{t=-\infty}^{s-1} g(t,x(t),\Delta x(t))\right)}{1-\sum\limits_{i=-\infty}^{+\infty}\gamma_i}$$

$$+ \sum_{s=-\infty}^{n-1}\frac{1}{\Psi^{-1}(q(s))}\Psi^{-1}\left(B_g(x) - \sum_{t=-\infty}^{s-1} g(t,x(t),\Delta x(t))\right)$$

where $B_g(x) \in \left[0, \sum\limits_{s=-\infty}^{+\infty} g(s, x(s), \Delta x(s))\right]$ satisfying

$$\frac{1-\sum\limits_{i=-\infty}^{+\infty}\gamma_i}{1-\sum\limits_{i=-\infty}^{+\infty}\delta_i}\sum_{i=-\infty}^{+\infty}\gamma_i\sum_{s=-\infty}^{i-1}\frac{1}{\Psi^{-1}(q(s))}\Psi^{-1}\left(B_g(x)-\sum_{t=-\infty}^{s-1}g(t,x(t),\Delta x(t))\right)$$

$$+\sum_{s=-\infty}^{+\infty}\frac{1}{\Psi^{-1}(q(s))}\Psi^{-1}\left(B_g(x)-\sum_{t=-\infty}^{s-1}g(t,x(t),\Delta x(t))\right)$$

$$=\sum_{i=-\infty}^{+\infty}\delta_i\sum_{s=-\infty}^{i-1}\frac{1}{\Psi^{-1}(q(s))}\Psi^{-1}\left(B_g(x)-\sum_{t=-\infty}^{s-1}g(t,x(t),\Delta x(t))\right).$$

Lemma 15. *Suppose that (b), (c), (d) and (e) hold. Then*

(i): *it holds that*

$$\begin{cases} \Delta[p(n)\Phi(\Delta(T_1y)(n))] + f(n, y(n), \Delta y(n)) = 0, \quad n \in \mathbf{Z}, \\ \Delta[q(n)\Psi(\Delta(T_2x)(n))] + g(n, x(n), \Delta x(n)) = 0, \quad n \in \mathbf{Z}, \\ \lim\limits_{n\to-\infty}(T_1y)(n) - \sum\limits_{n=-\infty}^{+\infty}\alpha_n(T_1y)(n) = 0, \\ \lim\limits_{n\to-\infty}(T_2x)(n) - \sum\limits_{n=-\infty}^{+\infty}\gamma_n(T_2x)(n) = 0, \\ \lim\limits_{n\to+\infty}(T_1y)(n) - \sum\limits_{n=-\infty}^{+\infty}\beta_n(T_1y)(n) = 0, \\ \lim\limits_{n\to+\infty}(T_2x)(n) - \sum\limits_{n=-\infty}^{+\infty}\delta_n(T_2x)(n) = 0; \end{cases}$$

(ii): $T(x, y) \in P$ *for each* $(x, y) \in P$*;*
(iii): (x, y) *is a positive solution of BVP(5) if and only if* $(x, y) \in P$ *is a solution of the operator equation* $(x, y) = T(x, y)$*;*
(iv): $T : P \to P$ *is completely continuous.*

Proof. For(i), (ii) and (iii), the proofs follow from Lemmas 2, 3, 4, 5, 6 and 7.

(iv) It suffices to prove that T is continuous on P and T maps bounded subsets into relatively compact sets. We divide the proof into four steps:

Step 1: Prove that both $y \to A_f(y)$ and $x \to B_g(x)$ are continuous.

Let $(x_k, y_k) \in P$ with $y_k \to y_0$ and $x_k \to x_0$ as $k \to +\infty$. Then there exists positive number $r > 0$ such that

$$\sup_{n\in Z} x_k(n),\ \sup_{n\in Z} y_k(n),\ \sup_{n\in Z} \Phi^{-1}(p(n))|\Delta x_k(n)|,\ \sup_{n\in Z} \Psi^{-1}(q(n))|\Delta y_k(n)| \le r$$

for all $k = 0, 1, 2, \cdots$. Hence there exists a bilateral nonnegative sequence $\{\phi_r(n)\}$ with $\sum_{n=-\infty}^{+\infty} \phi_r(n) + \infty$ satisfying

$$0 \le f(n, y_k(n), \Delta y_k(n)) = f\left(n, y_k(n), \frac{1}{\Psi^{-1}(q(n))}\Psi^{-1}(q(n))\Delta y_k(n)\right) \le \phi_r(n),$$

and

$$0 \le g(n, x_k(n), \Delta x_k(n)) = g\left(n, x_k(n), \frac{1}{\Phi^{-1}(p(n))}\Phi^{-1}(p(n))\Delta x_k(n)\right) \le \phi_r(n).$$

One sees that

$$0 \le A_f(y_k) \le \sum_{t=-\infty}^{+\infty} f(t, y_k(t), \Delta y_k(t)) \le \sum_{t=-\infty}^{+\infty} \phi_r(t).$$

We need to prove that $A_f(y_k) \to A_f(y_0)$ as $k \to +\infty$. If $A_f(y_k) \not\to A_f(y_0)$ as $k \to +\infty$, then there exist a sub-sequence such that $A_f(y_{ki})^1 \to a_1 \ne A_f(y_0)$ as $i \to +\infty$. Then

$$\frac{1-\sum_{i=-\infty}^{+\infty}\alpha_i}{1-\sum_{i=-\infty}^{+\infty}\beta_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y_{ki})^1 - \sum_{t=-\infty}^{s-1} f(t, y_{ki}^1(t), \Delta y_{ki}^1(t))\right)$$

$$+\sum_{s=-\infty}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y_{ki})^1 - \sum_{t=-\infty}^{s-1} f(t, y_{ki}^1(t), \Delta y_{ki}^1(t))\right)$$

$$= \sum_{i=-\infty}^{+\infty}\beta_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y_{ki})^1 - \sum_{t=-\infty}^{s-1} f(t, y_{ki}^1(t), \Delta y_{ki}^1(t))\right).$$

Let $i \to +\infty$, by using the generalized Leibegue dominated convergence theorem, we get that

$$\frac{1-\sum_{i=-\infty}^{+\infty}\alpha_i}{1-\sum_{i=-\infty}^{+\infty}\beta_i}\sum_{i=-\infty}^{+\infty}\alpha_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(a_1-\sum_{t=-\infty}^{s-1}f(t,y_0(t),\Delta y_0(t))\right)$$

$$+\sum_{s=-\infty}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(a_1-\sum_{t=-\infty}^{s-1}f(t,y_0(t),\Delta y_0(t))\right)$$

$$=\sum_{i=-\infty}^{+\infty}\beta_i\sum_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(a_1-\sum_{t=-\infty}^{s-1}f(t,y_0(t),\Delta y_0(t))\right).$$

From Lemma 2, we know that $a_1 = A_f(y_0)$, a contradiction. Hence $A_f(y_k) \to A_f(y_0)$ as $k \to +\infty$. Similarly we can prove that $B_g(x_k) \to B_g(x_0)$ as $k \to +\infty$.

Step 2: Prove that both $T_1 : Y \to X$ and $T_2 : X \to Y$ are continuous.

Let $(x_k, y_k) \in P$ with $y_k \to y_0$ and $x_k \to x_0$ as $k \to +\infty$. We need to prove that $T_1 y_k \to T_1 y_0$ as $k \to +\infty$ and $T_2 x_k \to T_2 x_0$ as $k \to +\infty$. By Step 1, $A_f(y_k) \to A_f(y_0$ and $B_g(y_k) \to B_g(x_0)$ as $k \to +\infty$. This together with the continuous property of f, g implies that T is continuous at (x_0, y_0).

Step 3: For each bounded subset $\Omega \subset P$, prove that $T\Omega$ is bounded.

Since $\Omega \subset P$ is bounded, then there exists positive number $r > 0$ such that

$$\sup_{n\in\mathbf{Z}} x(n), \sup_{n\in\mathbf{Z}} y(n), \sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y(n)| \le r$$

for all $(x, y) \in \Omega$ Hence there exists a bilateral nonnegative sequence $\{\phi_r(n)\}$ with $\sum_{n=-\infty}^{+\infty}\phi_r(n) + \infty$ satisfying

$$0 \le f(n, y(n), \Delta y(n)) \le \phi_r(n),$$

and

$$0 \le g(n, x(n), \Delta x(n)) \le \phi_r(n).$$

One sees that

$$0 \le A_f(y) \le \sum_{t=-\infty}^{+\infty} f(t, y(t), \Delta y(t)) \le \sum_{t=-\infty}^{+\infty}\phi_r(t) =: M_0$$

and

$$0 \le B_g(x) \le \sum_{t=-\infty}^{+\infty} g(t, x(t), \Delta x(t)) \le \sum_{t=-\infty}^{+\infty} \phi_r(t) = M_0.$$

Then

$$(T_1 y)(n) = \frac{\sum\limits_{i=-\infty}^{+\infty} \alpha_i \sum\limits_{s=-\infty}^{i-1} \frac{1}{\Phi^{-1}(p(s))} \Phi^{-1}\left(A_f(y) - \sum\limits_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t))\right)}{1 - \sum_{i=-\infty}^{+\infty} \alpha_i}$$

$$+ \sum_{s=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(s))} \Phi^{-1}\left(A_f(y) - \sum_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t))\right)$$

$$\le \frac{\sum\limits_{i=-\infty}^{+\infty} \alpha_i \sum\limits_{s=-\infty}^{i-1} \frac{1}{\Phi^{-1}(p(s))} \Phi^{-1}\left(\sum\limits_{t=-\infty}^{+\infty} f(t, y(t), \Delta y(t)) - \sum\limits_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t))\right)}{1 - \sum_{i=-\infty}^{+\infty} \alpha_i}$$

$$+ \sum_{s=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(s))} \Phi^{-1}\left(\sum_{t=-\infty}^{+\infty} f(t, y(t), \Delta y(t)) - \sum_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t))\right)$$

$$\le \frac{\Phi^{-1}\left(\sum\limits_{t=-\infty}^{+\infty} \phi_r(t)\right) \sum\limits_{i=-\infty}^{+\infty} \alpha_i \sum\limits_{s=-\infty}^{i-1} \frac{1}{\Phi^{-1}(p(s))}}{1 - \sum_{i=-\infty}^{+\infty} \alpha_i} + \Phi^{-1}\left(\sum_{t=-\infty}^{+\infty} \phi_r(t)\right) \sum_{s=-\infty}^{+\infty} \frac{1}{\Phi^{-1}(p(s))}$$

$$=: M_1,$$

and

$$\Phi^{-1}(p(n)) |\Delta(T_1 y)(n)| = \left| \Phi^{-1}\left(A_f(y) - \sum_{t=-\infty}^{s-1} f(t, y(t), \Delta y(t))\right) \right|$$

$$\le \Phi^{-1}\left(2 \sum_{t=-\infty}^{+\infty} \phi_r(t)\right) =: M_2.$$

Similarly, one has that

$$(T_2 x)(n) \le \frac{\Psi^{-1}\left(\sum\limits_{t=-\infty}^{+\infty} \phi_r(t)\right) \sum\limits_{i=-\infty}^{+\infty} \gamma_i \sum\limits_{s=-\infty}^{i-1} \frac{1}{\Psi^{-1}(q(s))}}{1 - \sum_{i=-\infty}^{+\infty} \gamma_i}$$

$$+ \Psi^{-1}\left(\sum_{t=-\infty}^{+\infty} \phi_r(t)\right) \sum_{s=-\infty}^{+\infty} \frac{1}{\Psi^{-1}(q(s))} =: M_3$$

and

$$\Psi^{-1}(q(n))|\Delta(T_2x)(n)| \leq \Psi^{-1}\left(2\sum_{t=-\infty}^{+\infty}\phi_r(t)\right) =: M_4.$$

It follows that $T\Omega$ is bounded.

Step 4: For each bounded subset $\Omega \subset P$, prove that $T\Omega$ is relatively compact.

Since $\Omega \subset P$ is bounded, then there exists positive number $r > 0$ such that

$$\sup_{n\in\mathbf{Z}} x(n), \sup_{n\in\mathbf{Z}} y(n), \sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y(n)| \leq r$$

for all $(x, y) \in \Omega$ Hence there exists a bilateral nonnegative sequence $\{\phi_r(n)\}$ with $\sum\limits_{n=-\infty}^{+\infty} \phi_r(n) + \infty$ satisfying

$$0 \leq f(n, y(n), \Delta y(n)) = f\left(n, y(n), \frac{1}{\Psi^{-1}(q(n))}\Psi^{-1}(q(n))\Delta y(n)\right) \leq \phi_r(n),$$

and

$$0 \leq g(n, x(n), \Delta x(n)) = g\left(n, x(n), \frac{1}{\Phi^{-1}(p(n))}\Phi^{-1}(p(n))\Delta x(n)\right) \leq \phi_r(n).$$

One sees that

$$\begin{aligned} &0 \leq A_f(y) \leq \sum_{t=-\infty}^{+\infty} f(t, y(t), \Delta y(t)) \leq \sum_{t=-\infty}^{+\infty} \phi_r(t) =: M_0, \\ (39) \qquad &0 \leq B_g(x) \leq \sum_{t=-\infty}^{+\infty} g(t, x(t), \Delta x(t)) \leq \sum_{t=-\infty}^{+\infty} \phi_r(t) = M_0. \end{aligned}$$

Then

$$\left|(T_1y)(n) - \frac{\sum\limits_{i=-\infty}^{+\infty}\alpha_i\sum\limits_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y)-\sum\limits_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))\right)}{1-\sum_{i=-\infty}^{+\infty}\alpha_i}\right|$$

$$\leq \sum_{s=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))\right)$$

$$\leq \sum_{s=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum_{t=-\infty}^{+\infty}\phi_r(t)\right) = \Phi^{-1}\left(\sum_{t=-\infty}^{+\infty}\phi_r(t)\right)\sum_{s=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(s))}$$

$\to 0$ uniformly as $n \to -\infty$,

$$\left|(T_1y)(n) - \frac{\sum\limits_{i=-\infty}^{+\infty}\alpha_i\sum\limits_{s=-\infty}^{i-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y)-\sum\limits_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))\right)}{1-\sum_{i=-\infty}^{+\infty}\alpha_i}\right.$$

$$\left.-\sum_{s=-\infty}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))\right)\right|$$

$$\leq \sum_{s=n}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))\right)$$

$$\leq \sum_{s=n}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum_{t=-\infty}^{+\infty}\phi_r(t)\right) = \Phi^{-1}\left(\sum_{t=-\infty}^{+\infty}\phi_r(t)\right)\sum_{s=n}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}$$

$\to 0$ uniformly as $n \to +\infty$.

On the other hand, we have

$$|\Phi^{-1}(p(n))\Delta(T_1y)(n) - \Phi^{-1}(A_f(y))| =$$

$$\left|\Phi^{-1}\left(A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))\right)-\Phi^{-1}(A_f(y))\right|$$

and

$$\left|\Phi^{-1}(p(n))\Delta(T_1y)(n)-\Phi^{-1}\left(A_f(y)-\sum_{t=-\infty}^{+\infty}f(t,y(t),\Delta y(t))\right)\right|=$$

$$\left|\Phi^{-1}\left(A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))\right)\right.$$

$$\left.-\Phi^{-1}\left(A_f(y)-\sum_{t=-\infty}^{+\infty}f(t,y(t),\Delta y(t))\right)\right|.$$

For any $\epsilon>0$, since Φ^{-1} is uniformly continuous on $[-2M_0,2M_0]$, then there exists $\lambda>0$ such that $|\Phi^{-1}(u_1)-\Phi^{-1}(u_2)|<\epsilon$ for all $u_1,u_2\in[-2M_0,2M_0]$ with $|u_1-u_2|<\lambda$.

From

$$\left|A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))-A_f(y)\right|\le\sum_{t=-\infty}^{s-1}\phi_r(t),$$

$$\left|A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))-\left(A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))\right)\right|$$

$$\le\sum_{t=s}^{+\infty}\phi_r(t),$$

we know that there exists $N>0$ such that

$$\left|A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))-A_f(y)\right|<\lambda \text{ uniformly as } s<-N,$$

$$\left|A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))-\left(A_f(y)-\sum_{t=-\infty}^{s-1}f(t,y(t),\Delta y(t))\right)\right|$$

$$<\lambda \text{ uniformly as } s>N.$$

It follows that

$$|\Phi^{-1}(p(n))\Delta(T_1y)(n)-\Phi^{-1}(A_f(y))|<\epsilon \text{ uniformly as } s<-N,$$

$$\left|\Phi^{-1}(p(n))\Delta(T_1y)(n)-\Phi^{-1}\left(A_f(y)-\sum_{t=-\infty}^{+\infty}f(t,y(t),\Delta y(t))\right)\right|$$

$$<\epsilon \text{ uniformly as } s>N.$$

Then

$(T_1y)(n)$ is uniformly convergent as $n \to -\infty$,

$(T_1y)(n)$ is uniformly convergent as $n \to +\infty$,

$\Phi^{-1}(p(n))\Delta(T_1y)(n)$ is uniformly convergent as $n \to -\infty$,

$\Phi^{-1}(p(n))\Delta(T_1y)(n)$ is uniformly convergent as $n \to +\infty$.

Similarly, one has that

$(T_2x)(n)$ is uniformly convergent as $n \to -\infty$,

$(T_2x)(n)$ is uniformly convergent as $n \to +\infty$,

$\Psi^{-1}(q(n))\Delta(T_2x)(n)$ is uniformly convergent as $n \to -\infty$,

$\Psi^{-1}(q(n))\Delta(T_2x)(n)$ is uniformly convergent as $n \to +\infty$.

One knows that $T\Omega$ is relatively compact. Steps 1, 2, 3 and 4 imply that T is completely continuous.

□

Now, we establish existence of three positive solutions of BVP(5) by using Lemma 1. Denote

$$M = \max\left\{ \frac{\sum\limits_{t=-\infty}^{+\infty} \frac{1}{\Phi^{-1}(p(t))} + 2\sum\limits_{t=-\infty}^{+\infty} \alpha_t \sum\limits_{s=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(s))}}{1 - \sum\limits_{n=-\infty}^{+\infty} \alpha_n}, \right.$$

$$\left. \frac{\sum\limits_{t=-\infty}^{+\infty} \frac{1}{\Psi^{-1}(q(t))} + 2\sum\limits_{t=-\infty}^{+\infty} \gamma_t \sum\limits_{s=-\infty}^{n-1} \frac{1}{\Psi^{-1}(q(s))}}{1 - \sum\limits_{n=-\infty}^{+\infty} \gamma_n} \right\}$$

and for positive numbers e_1, e_2, c and integers k_1, k_2, denote

$$W = \min\left\{ \Phi\left(\frac{e_2}{\mu \sum\limits_{s=\left[\frac{k_1+k_2}{2}\right]+1}^{k_2} \frac{1}{\Phi^{-1}(p(s))} \Phi^{-1}\left(\sum\limits_{t=\left[\frac{k_1+k_2}{2}\right]+1}^{s-1} 2^{-|t|} \right)} \right), \right.$$

$$\Phi\left(\frac{e_2}{\mu \sum\limits_{s=k_1}^{\left[\frac{k_1+k_2}{2}\right]} \frac{1}{\Phi^{-1}(p(s))} \Phi^{-1}\left(\sum\limits_{t=s}^{\left[\frac{k_1+k_2}{2}\right]} 2^{-|t|} \right)} \right)$$

$$\Psi\left(\frac{e_2}{\mu \sum\limits_{s=\left[\frac{k_1+k_2}{2}\right]+1}^{k_2} \frac{1}{\Psi^{-1}(q(s))} \Psi^{-1}\left(\sum\limits_{t=\left[\frac{k_1+k_2}{2}\right]+1}^{s-1} 2^{-|t|} \right)} \right),$$

$$\left. \Psi\left(\frac{e_2}{\mu \sum\limits_{s=k_1}^{\left[\frac{k_1+k_2}{2}\right]} \frac{1}{\Psi^{-1}(q(s))} \Psi^{-1}\left(\sum\limits_{t=s}^{\left[\frac{k_1+k_2}{2}\right]} 2^{-|t|} \right)} \right) \right\};$$

$$Q = \max\left\{ \frac{\Phi(c)}{6},\ \frac{\Psi(c)}{6},\ \frac{1}{6}\Phi\left(\frac{c}{M}\right),\ \frac{1}{6}\Psi\left(\frac{c}{M}\right) \right\},$$

$$E = \max\left\{ \frac{1}{6}\Phi\left(\frac{e_1}{M}\right),\ \frac{1}{6}\Psi\left(\frac{e_1}{M}\right) \right\}.$$

THEOREM 2. *Choose* $k_1, k_2 \in N$ *with* $k_1+3 < k_2$. *Let* $\mu = \min\{\mu_1, \mu_2\}$ *with* μ_1, μ_2 *being defined by (36) and (38). Suppose that (b)-(e) hold and there exist positive constants* e_1, e_2, c *such that*

$$c \geq \frac{e_2}{\mu} > e_2 > e_1 > 0.$$

If $Q > W$ *and*

(A1): $\begin{array}{l} f\left(n,u,\frac{v}{\psi^{-1}(q(n))}\right) \le \frac{Q}{2^{|n|}} \\ g\left(n,u,\frac{v}{\phi^{-1}(p(n))}\right) \le \frac{Q}{2^{|n|}} \end{array}$ *for all* $n \in \mathbf{Z}, u \in [0,c], v \in [0,c]$;

(A2): $\begin{array}{l} f\left(n,u,\frac{v}{\psi^{-1}(q(n))}\right) \ge \frac{W}{2^{|n|}} \\ g\left(n,u,\frac{v}{\phi^{-1}(p(n))}\right) \ge \frac{W}{2^{|n|}} \end{array}$ *for all* $n \in [k_1,k_2]$, $u \in [e_2, \frac{e_2}{\mu}], v \in [0,c]$;

(A3): $\begin{array}{l} f\left(n,u,\frac{v}{\psi^{-1}(q(n))}\right) \le \frac{E}{2^{|n|}} \\ g\left(n,u,\frac{v}{\phi^{-1}(p(n))}\right) \le \frac{E}{2^{|n|}} \end{array}$ *for all* $n \in \mathbf{Z}, u \in [0,e_1], v \in [0,c]$.

Then BVP(5) has at least three positive solutions $(x_1,y_1),(x_2,y_2)$ *and* (x_3,y_3) *such that*

$$\sup_{n\in\mathbf{Z}} x_1(n),\ \sup_{n\in\mathbf{Z}} y_1(n) < e_1,\ \min_{n\in[k_1,k_2]} x_2(n),\ \min_{n\in[k_1,k_2]} y_2(n) > e_2, \tag{40}$$

and

$$\text{either } \sup_{n\in\mathbf{Z}} x_3(n) \text{ or } \sup_{n\in\mathbf{Z}} y_3(n) > e_1, \tag{41}$$

$$\text{either } \min_{n\in[k_1,k_2]} x_3(n) \text{ or } \min_{n\in[k_1,k_2]} y_3(n) < e_2. \tag{42}$$

Proof. Let E, P and T be defined in Section 2. We complete the proof of Theorem 1 by using Lemma 1. Define the following functionals by

$$\begin{aligned}
\gamma(x,y) &= \max\left\{\sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x(n)|,\ \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y(n)|\right\},\\
\beta(x,y) &= \max\left\{\sup_{n\in\mathbf{Z}} x(n),\ \sup_{n\in\mathbf{Z}} y(n)\right\}, (x,y) \in P,\\
\theta(x,y) &= \max\left\{\sup_{n\in\mathbf{Z}} x(n),\ \sup_{n\in\mathbf{Z}} y(n)\right\}, (x,y) \in P,\\
\alpha(x,y) &= \min\left\{\min_{n\in[k_1,k_2]} x(n),\ \min_{n\in[k_1,k_2]} y(n)\right\}, (x,y) \in P,\\
\varphi(x,y) &= \min\left\{\min_{n\in[k_1,k_2]} x(n),\ \min_{n\in[k_1,k_2]} y(n)\right\}, (x,y) \in P.
\end{aligned}$$

It is easy to see that α, ψ are two nonnegative continuous concave functionals on the cone P, γ, β, θ are three nonnegative continuous convex functionals on the cone P.

One sees $\alpha(x,y) \le \beta(x,y)$ for all $(x,y) \in P$. Lemmas in Section 2 imply that $(x,y) = (x(n),y(n))_{n=-\infty}^{+\infty}$ is a positive solution of BVP(5) if and only if (x,y) is a solution of the operator equation $(x,y) = T(x,y)$ and $T : P \to P$ is completely continuous.

By the definition of P, for $(x,y) \in P$, we have

$$\begin{aligned} & x(n), y(n) \ge 0 \text{ for all } n \in \mathbf{Z}, \\ & \lim_{n\to-\infty} x(n) - \sum_{n=-\infty}^{+\infty} \alpha_n x(n) = 0, \\ & \lim_{n\to-\infty} y(n) - \sum_{n=-\infty}^{+\infty} \gamma_n y(n) = 0. \end{aligned}$$

Then

$$0 \le x(n) = \frac{x(n) - \lim\limits_{n\to-\infty} x(n) + \lim\limits_{n\to-\infty} x(n) - x(n) \sum\limits_{n=-\infty}^{+\infty} \alpha_n}{1 - \sum\limits_{n=-\infty}^{+\infty} \alpha_n}$$

$$= \frac{\sum\limits_{t=-\infty}^{n-1} \Delta x(t) + \sum\limits_{t=-\infty}^{+\infty} \alpha_t x(t) - x(n) \sum\limits_{t=-\infty}^{+\infty} \alpha_t}{1 - \sum\limits_{n=-\infty}^{+\infty} \alpha_n}$$

$$= \frac{\sum\limits_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} \Phi^{-1}(p(t)) \Delta x(t) + \sum\limits_{t=-\infty}^{n-1} \alpha_t [x(t) - x(n)] + \sum\limits_{t=n}^{+\infty} \alpha_t [x(t) - x(n)]}{1 - \sum\limits_{n=-\infty}^{+\infty} \alpha_n}$$

$$= \frac{\sum\limits_{t=-\infty}^{n-1} \frac{1}{\Phi^{-1}(p(t))} \Phi^{-1}(p(t)) \Delta x(t) - \sum\limits_{t=-\infty}^{n-1} \alpha_t \sum\limits_{s=t}^{n-1} \frac{1}{\Phi^{-1}(p(s))} \Phi^{-1}(p(s)) \Delta x(s)}{1 - \sum\limits_{n=-\infty}^{+\infty} \alpha_n}$$

$$+ \frac{\sum\limits_{t=n}^{+\infty} \alpha_t \sum\limits_{s=n}^{t-1} \frac{1}{\Phi^{-1}(p(s))} \Phi^{-1}(p(s)) \Delta x(s)}{1 - \sum\limits_{n=-\infty}^{+\infty} \alpha_n}$$

$$\le \frac{\sum\limits_{t=-\infty}^{+\infty} \frac{1}{\phi^{-1}(p(t))} + 2 \sum\limits_{t=-\infty}^{+\infty} \alpha_t \sum\limits_{s=-\infty}^{n-1} \frac{1}{\phi^{-1}(p(s))}}{1 - \sum\limits_{n=-\infty}^{+\infty} \alpha_n} \sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n)) |\Delta x(n)|$$

and

$$0 \le y(n) \le \frac{\sum\limits_{t=-\infty}^{+\infty} \frac{1}{\Psi^{-1}(q(t))} + 2\sum\limits_{t=-\infty}^{+\infty} \gamma_t \sum\limits_{s=-\infty}^{n-1} \frac{1}{\Psi^{-1}(q(s))}}{1 - \sum\limits_{n=-\infty}^{+\infty} \gamma_n} \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y(n)|.$$

we have $||(x,y)|| \le M\gamma(x,y)$ for all $(x,y) \in P$.

Corresponding to Lemma 1, choose

$$h = \mu e_1, \quad d = e_1, \quad a = e_2, \quad b = \frac{e_2}{\mu}, \quad c = c.$$

Now, we prove that all conditions of Lemma 1 hold. One sees that $0 < d < a$. The remainder is divided into five steps.

Step 1: Prove that $T : \overline{P_c} \to \overline{P_c}$;

For $(x,y) \in \overline{P_c}$, we have $||(x,y)|| \le c$. Then

$$0 \le x(n), y(n) \le c, n \in \mathbf{Z},$$
$$0 \le \Phi^{-1}(p(n))|\Delta x(n)|, \Psi^{-1}(q(n))|\Delta y(n)| \le c \text{ for } n \in \mathbf{Z}.$$

So (A1) implies that

$$f(n, y(n), \Delta y(n)) = f\left(n, y(n), \frac{\Psi^{-1}(q(n))\Delta y(n)}{\Psi^{-1}(q(n))}\right) \le \frac{Q}{2^{|n|}}, \ n \in \mathbf{Z},$$

$$g(n, x(n), \Delta x(n)) = g\left(n, x(n), \frac{\Phi^{-1}(p(n))\Delta x(n)}{\Phi^{-1}(p(n))}\right) \le \frac{Q}{2^{|n|}}, \ n \in \mathbf{Z}.$$

It follows from

$$0 \le A_f(y) \le \sum_{j=-\infty}^{+\infty} f(j, y(j), \Delta y(j)) \tag{43}$$

that

$$\begin{aligned}
\Phi^{-1}(p(n))|\Delta(T_1 y)(n)| &= \left|\Phi^{-1}\left(A_f(y) - \sum_{j=-\infty}^{n-1} f(j, y(j), \Delta y(j))\right)\right| \\
&\le \Phi^{-1}\left(2\sum_{j=-\infty}^{+\infty} f(j, y(j), \Delta y(j))\right) \\
&\le \Phi^{-1}\left(2\sum_{j=-\infty}^{+\infty} \frac{Q}{2^{|j|}}\right) \\
&\le \Phi^{-1}(6Q) \le c.
\end{aligned}$$

So

$$\sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta(T_1y)(n)| \leq c. \tag{44}$$

Then $T(x,y) \in P$ implies that

$$|(T_1y)(n)| \leq M \sup_{n\in\mathbf{Z}} \phi^{-1}(p(n))|\Delta(T_1y)(n)| \leq M\Phi^{-1}(6Q) \leq c.$$

Hence

$$\sup_{n\in\mathbf{Z}} |(T_1y)(n)| \leq c. \tag{45}$$

Similarly we can show that

$$\sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta|(T_2x)(n))| \leq c, \ \sup_{n\in\mathbf{Z}} |(T_2x)(n)| \leq c. \tag{46}$$

It follows from (24)-(26) that $||T(x,y)|| \leq c$. Then $T: \overline{P_c} \to \overline{P_c}$.
Step 2: Prove that

$$\{(x,y) \in P(\gamma,\theta,\alpha;a,b,c)|\alpha(x,y) > a\} =$$

$$\left\{(x,y) \in P\left(\gamma,\theta,\alpha;e_2,\tfrac{e_2}{\mu},c\right)|\alpha(x,y) > e_2\right\} \neq \emptyset$$

and $\alpha(T(x,y)) > e_2$ for every $(x,y) \in P\left(\gamma,\theta,\alpha;e_2,\frac{e_2}{\mu},c\right)$;

By the definition of μ, we can choose $A_i(i=1,2)$ and $k_1 < l < l+1 < k_2$ such that

$$A_1 \in \left(e_2, \frac{e_2}{\mu}\right], \ A_2 \in \left(e_2, \frac{e_2}{\mu}\right].$$

Since $\mu \in (0,1)$, $1 > \sum_{n=-\infty}^{+\infty} \alpha_n$ and $1 > \sum_{n=-\infty}^{+\infty} \gamma_n$, we can choose

$$\min\left\{\tfrac{e_2}{\mu}, \tfrac{c}{\Psi^{-1}(q(k_2))}\right\} \geq D_2 \geq A_2 > e_2,$$

$$\min\left\{\tfrac{e_2}{\mu}, \tfrac{c}{\Phi^{-1}(p(k_2))}\right\} \geq D_1 \geq A_1 > e_2$$

and $B_i(i=1,2)$ satisfying

$$\mu D_1 \sum_{n\in[l+1,k_2]} \alpha_n \leq \left(1 - \sum_{n\notin[k_1,k_2]} \alpha_n - \sum_{n\in[k_1,l]} \alpha_n\right) A_1,$$

$$B_1 = \frac{A_1 \sum_{n\in[k_1,l]} \alpha_n + D_1 \sum_{n\in[l+1,k_2]} \alpha_n}{1 - \sum_{n\notin[k_1,k_2]} \alpha_n},$$

$$|D_1 - A_1| \leq \frac{c}{\Phi^{-1}(p(l))},$$

$$\mu D_2 \sum_{n\in[l+1,k_2]} \gamma_n \leq \left(1 - \sum_{n\notin[k_1,k_2]} \gamma_n - \sum_{n\in[k_1,l]} \gamma_n\right) A_2,$$

$$B_2 = \frac{A_2 \sum_{n\in[k_1,l]} \gamma_n + D_2 \sum_{n\in[l+1,k_2]} \gamma_n}{1 - \sum_{n\notin[k_1,k_2]} \gamma_n},$$

$$|D_2 - A_2| \leq \frac{c}{\Psi^{-1}(q(l))}.$$

It is easy to show that $B_i \leq D_i(i=1,2)$ and $|B_i - D_i| \leq D_i$. Let

$$x(n) = \begin{cases} A_1, n \in [k_1, l], \\ D_1, n \in [l+1, k_2], \\ B_1, n \notin [k_1, k_2], \end{cases} \quad y(n) = \begin{cases} A_2, n \in [k_1, l], \\ D_2, n \in [l+1, k_2], \\ B_2, n \notin [k_1, k_2]. \end{cases}$$

It is easy to show that $(x,y) \in P$ and

$$\min_{n\in[k_1,k_2]} x(n) = \min\{A_1, D_1\} = A_1 > e_2,$$

$$\min_{n\in[k_1,k_2]} y(n) = \min\{A_2, D_2\} = A_2 > e_2,$$

$$\sup_{n\in\mathbf{Z}} x(n) \leq \max\{A_1, B_1, D_1\} = \max\{B_1, D_1\} = D_1 \leq \frac{e_2}{\mu} = b,$$

$$\sup_{n\in\mathbf{Z}} y(n) \leq \max\{A_2, B_2, D_2\} = \max\{B_2, D_2\} = D_2 \leq \frac{e_2}{\mu} = b,$$

$$\sup_{n\in\mathbf{Z}}\Phi^{-1}(p(n))|\Delta x(n)| = \max\{\Phi^{-1}(p(k_1-1))|A_1-B_1|,\\ \Phi^{-1}(p(k_2))|B_1-D_1|, \Phi^{-1}(p(l))|A_1-D_1|\} \le c,$$

$$\sup_{n\in\mathbf{Z}}\Psi^{-1}(q(n))|\Delta y(n)| = \max\{\Psi^{-1}(q(k_1-1))|A_2-B_2|,\\ \Psi^{-1}(q(k_2))|B_2-D_2|, \Psi^{-1}(q(l))|A_2-D_2|\} \le c.$$

Then

$$\alpha(x,y) > e_2,\ \theta(x,y) \le b,\ \gamma(x,y) \le c.$$

It follows that $\{(x,y) \in P(\gamma,\theta,\alpha;a,b,c)|\alpha(x,y) > a\} \neq \emptyset$.

For $(x,y) \in P(\gamma,\theta,\alpha;a,b,c)$, one has that

$$\alpha(x,y) = \min\left\{\min_{n\in[k_1,k_2]} x(n),\ \min_{n\in[k_1,k_2]} y(n)\right\} \ge e_2,$$

$$\theta(x) = \max\left\{\sup_{n\in\mathbf{Z}} x(n), \sup_{n\in\mathbf{Z}} y(n)\right\} \le \frac{e_2}{\mu},$$

and

$$\gamma(x) = \max\left\{\sup_{n\in\mathbf{Z}}\Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}}\Psi^{-1}(q(n))|\Delta y(n)|\right\} \le c.$$

Then

$$e_2 \le x(n), y(n) \le \frac{e_2}{\mu},\ n \in [k_1,k_2],$$

and

$$0 \le \Phi^{-1}(p(n))|\Delta x(n)|, \Psi^{-1}(q(n))|\Delta y(n)| \le c.$$

Thus (A2) implies that

$$f(n,y(n),\Delta y(n)), g(n,x(n),\Delta x(n)) \ge \frac{W}{2^{|n|}},\ \ n \in [k_1,k_2].$$

Then from the methods used in Lemma 3, we have $(T_1y)(n) > 0$ for all $n \in \mathbf{Z}$ and there exists $n_2 \in \mathbf{Z}$ such that $\Phi^{-1}(p(n))\Delta(T_1y)(n) \ge 0$ for all $n \le n_2$ and $\Phi^{-1}(p(n))\Delta(T_1y)(n) \le 0$ for all $n \ge n_2+1$,

and

$$(T_1y)(n) = \begin{cases} \lim\limits_{n\to-\infty}(T_1y)(n)+ \\ \sum\limits_{s=-\infty}^{n-1}\frac{\Phi^{-1}\left(\Phi^{-1}(p(n_2))\Delta(T_1y)(n_2)+\sum\limits_{t=s}^{n_2-1}f(t,y(t),\Delta y(t))\right)}{\Phi^{-1}(p(s))}, n\le n_2+1, \\ \lim\limits_{n\to+\infty}(T_1y)(n)- \\ \sum\limits_{s=n}^{+\infty}\frac{\Phi^{-1}\left(\Phi^{-1}(p(n_2+1))\Delta(T_1y)(n_2+1)-\sum\limits_{t=n_2+1}^{s-1}f(t,y(t),\Delta y(t))\right)}{\Phi^{-1}(p(s))}, n\ge n_2+1 \end{cases}$$

$$\ge \begin{cases} \sum\limits_{s=-\infty}^{n-1}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum\limits_{t=s}^{n_2-1}f(t,y(t),\Delta y(t))\right), n\le n_2+1, \\ \sum\limits_{s=n}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum\limits_{t=n_2+1}^{s-1}f(t,y(t),\Delta y(t))\right), n\ge n_2+1. \end{cases}$$

It is easy to see from $T(x,y)\in P$ that $\min_{n\in[k_1,k_2]}(T_1y)(n)\ge \mu\sup_{n\in Z}(T_1y)(n)=\mu(T_1y)(n_2+1)$. One sees that

$$(T_1y)(n_2+1)\ge \begin{cases} \sum\limits_{s=-\infty}^{n_2}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum\limits_{t=s}^{n_2-1}f(t,y(t),\Delta y(t))\right), \\ \sum\limits_{s=n_2+1}^{+\infty}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum\limits_{t=n_2+1}^{s-1}f(t,y(t),\Delta y(t))\right) \end{cases}$$

$$\ge \begin{cases} \sum\limits_{s=k_1}^{n_2}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum\limits_{t=s}^{n_2-1}f(t,y(t),\Delta y(t))\right), \\ \sum\limits_{s=n_2+1}^{k_2}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum\limits_{t=n_2+1}^{s-1}f(t,y(t),\Delta y(t))\right). \end{cases}$$

We consider two cases:

(i): $n_2>\left[\frac{k_1+k_2}{2}\right]$. We have

$$\min_{n\in[k_1,k_2]}(T_1y)(n)\ge\mu\sup_{n\in\mathbf{Z}}(T_1y)(n)=\mu(T_1y)(n_2+1)$$

$$\ge\mu\sum_{s=k_1}^{n_2}\frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum_{t=s}^{n_2-1}f(t,y(t),\Delta y(t))\right)$$

$$\geq \mu \sum_{s=k_1}^{\left[\frac{k_1+k_2}{2}\right]} \frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum_{t=s}^{\left[\frac{k_1+k_2}{2}\right]} f(t,y(t),\Delta y(t))\right)$$

$$\geq \mu \sum_{s=k_1}^{\left[\frac{k_1+k_2}{2}\right]} \frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum_{t=s}^{\left[\frac{k_1+k_2}{2}\right]} \frac{W}{2^{|t|}}\right) > e_2.$$

(ii): $n_2 \leq \left[\frac{k_1+k_2}{2}\right]$ We have

$$\min_{n\in[k_1,k_2]}(T_1y)(n) \geq \mu \sup_{n\in\mathbf{Z}}(T_1y)(n) = \mu(T_1y)(n_2+1)$$

$$\geq \mu \sum_{s=n_2+1}^{k_2} \frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum_{t=n_2+1}^{s-1} f(t,y(t),\Delta y(t))\right)$$

$$\geq \mu \sum_{s=\left[\frac{k_1+k_2}{2}\right]+1}^{k_2} \frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum_{t=\left[\frac{k_1+k_2}{2}\right]+1}^{s-1} f(t,y(t),\Delta y(t))\right)$$

$$\geq \mu \sum_{s=\left[\frac{k_1+k_2}{2}\right]+1}^{k_2} \frac{1}{\Phi^{-1}(p(s))}\Phi^{-1}\left(\sum_{t=\left[\frac{k_1+k_2}{2}\right]+1}^{s-1} \frac{W}{2^{|t|}}\right) > e_2.$$

So

$$\min_{n\in[k_1,k_2]}(T_1y)(n) > e_2.$$

Similarly we can show that $\min_{n\in[k_1,k_2]}(T_2x)(n) > e_2$. Then

$$\alpha(T(x,y)) = \min\left\{\min_{n\in[k_1,k_2]}(T_1y)(n), \min_{n\in[k_1,k_2]}(T_2x)(n)\right\} > e_2.$$

This completes Step 2.

Step 3: Prove that

$$\{(x,y)\in Q(\gamma,\theta,\varphi;h,d,c)|\beta(x,y)<d\} =$$

$$\{(x,y)\in Q\left(\gamma,\theta,\varphi;\mu e_1,e_1,c\right)|\beta(x,y)<e_1\} \neq \emptyset$$

and

$\beta(T(x,y)) < e_1$ for every $(x,y)\in Q(\gamma,\theta,\varphi;h,d,c) = Q\left(\gamma,\theta,\varphi;\mu e_1,e_1,c\right);$

Similarly to Step 2, we can see that $\{(x,y) \in Q(\gamma,\theta,\varphi;h,d,c) | \beta(x,y) < d\} \neq \emptyset$.

For $(x,y) \in Q(\gamma,\theta,\varphi;h,d,c)$, one has that

$$\varphi(x,y) = \min\left\{\min_{n\in[k_1,k_2]} x(n), \min_{n\in[k_1,k_2]} y(n)\right\} \geq \mu e_1$$

$$\theta(x,y) = \max\left\{\sup_{n\in\mathbf{Z}} x(n), \sup_{n\in\mathbf{Z}} y(n)\right\} \leq d = e_1,$$

and

$$\gamma(x,y) = \max\left\{\sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y(n)|\right\} \leq c.$$

Hence we get that

$$0 \leq x(n), y(n) \leq e_1, 0 \leq \Phi^{-1}(p(n))|\Delta x(n)|, \Psi^{-1}(q(n))|\Delta y(n)| \leq c,\ n \in \mathbf{Z}.$$

Then (A3) implies that

$$f(n,y(n),\Delta y(n)), g(n,x(n),\Delta x(n)) \leq \frac{E}{2^{|n|}},\ \ n \in \mathbf{Z}.$$

So (2.16) implies that

$$\begin{aligned}
&\beta(T(x,y)) \\
&\leq M \max\left\{\sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta(T_1 y)(n)|, \sup_{n\in Z} \Psi^{-1}(q(n))|\Delta(T_2 x)(n)|\right\} \\
&\leq M \max\left\{\Phi^{-1}(6E), \Psi^{-1}(6E)\right\} \leq e_1 = d.
\end{aligned}$$

This completes Step 3.

Step 4: Prove that $\alpha(T(x,y)) > a$ for $(x,y) \in P(\gamma,\alpha;a,c)$ with $\theta(T(x,y)) > b$;

For $(x,y) \in P(\gamma,\alpha;a,c) = P(\gamma,\alpha;e_2,c)$ with $\theta(T(x,y)) = \beta(T(x,y)) > b = \frac{e_2}{\mu}$, we have that

$$\begin{aligned}
\alpha(x,y) &= \min\left\{\min_{n\in[k_1,k_2]} x(n), \min_{n\in[k_1,k_2]} y(n)\right\} \geq e_2, \\
\gamma(x,y) &= \max\left\{\sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y(n)|\right\} \leq c, \\
&\sup_{n\in Z} (T(x,y))(n) > \frac{e_2}{\mu}.
\end{aligned}$$

Then

$$\alpha(T(x,y)) = \min_{n\in[k_1,k_2]} (T(x,y))(n) \geq \mu\beta(T(x,y)) > e_2 = a.$$

This completes Step 4.

Step 5: Prove that $\beta(T(x,y)) < d$ for each $(x,y) \in Q(\gamma,\beta;d,c)$ with $\varphi(Tx) < h$.

For $(x,y) \in Q(\gamma,\beta;d,c)$ with $\varphi(Tx) < h$, we have

$$\begin{aligned}
\gamma(x,y) &= \max\left\{\sup_{n\in\mathbf{Z}} \Phi^{-1}(p(n))|\Delta x(n)|, \sup_{n\in\mathbf{Z}} \Psi^{-1}(q(n))|\Delta y(n)|\right\} \leq c,\\
\beta(x,y) &= \max\left\{\sup_{n\in\mathbf{Z}} x(n), \sup_{n\in\mathbf{Z}} y(n)\right\} \leq d = e_1,\\
\varphi(T(x,y)) &= \min\left\{\min_{n\in[k_1,k_2]} (T_1 y)(n), \min_{n\in[k_1,k_2]} (T_2 x)(n)\right\} < h = \mu e_1.
\end{aligned}$$

Then

$$\beta(T(x,y)) \leq \frac{1}{\mu}\varphi(T(x,y)) < e_1 = d.$$

This completes the Step 5.

Then Lemma 1 implies that T has at least three fixed points (x_1,y_1), (x_2,y_2) and (x_3,y_3) such that

$$\beta(x_1,y_1) < e_1,\ \alpha(x_2,y_2) > e_2,\ \beta(x_3,y_3) > e_1,\ \alpha(x_3,y_3) < e_2.$$

Hence BVP(5) has three positive solutions $(x_1,y_1), (x_2,y_2)$ and (x_3,y_3) satisfying (40)-(42). The proof is complete. $\square$

5. An application

Now, we apply Theorem 2 to BVP(6). Corresponding to BVP(5), in (6) we have $p(n) = (|n|+1)^2 = q(n)$, $\Phi(x) = \Psi(x) = x$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = 0$ for every $n \in \mathbf{Z}$.

Note $\sum_{n=-\infty}^{+\infty} \frac{1}{(|n|+1)^2} = \frac{\pi^2}{3} - 1$ and $P_{+\infty} = 1 + \sum_{n=-\infty}^{+\infty} \frac{1}{(|n|+1)^2} = \frac{\pi^2}{3}$. Let $k_1 = 0$ and $k_2 = 102$. Denote

$$M_0 = \sum_{t=-\infty}^{+\infty} \frac{1}{p(t)} = \frac{\pi^2}{3} - 1,$$

$$\mu = \min\{\mu_1,\mu_2\} = \min\left\{\frac{P_{k_1} - P_{k_1-1}}{P_{+\infty} - P_{k_1-1}}, \frac{P_{k_2+1} - P_{k_2}}{P_{k_2+1}}\right\} \geq 0.004.$$

and for positive numbers e_1, e_2, c and integers k_1, k_2, denote

$$Q_0 = \max\left\{\frac{c}{6},\ \ \frac{1}{6}\frac{c}{\frac{\pi^2}{3}-1}\right\} = \frac{c}{6},\ E_0 = \frac{1}{6}\frac{e_1}{\frac{\pi^2}{3}-1}$$

$$W_0 = \min\left\{\frac{e_2}{\mu\sum\limits_{s=52}^{102}\frac{1}{(s+1)^2}\sum\limits_{t=52}^{s-1}2^{-|t|}}, \frac{e_2}{\mu\sum\limits_{s=0}^{51}\frac{1}{(s+1)^2}\sum\limits_{t=s}^{51}2^{-|t|}}\right\} \le 1308.078e_2.$$

THEOREM 3. *Suppose that there exist positive constants e_1, e_2, c such that $Q_0 > W_0$ and*

$$\textbf{(A4):}\quad \begin{array}{l} f\left(n,u,\frac{v}{(|n|+1)^2}\right) \le \frac{Q_0}{2^{|n|}}, n \in \mathbf{Z}, u \in [0,c], v \in [0,c], \\ g\left(n,u,\frac{v}{(|n|+1)^2}\right) \le \frac{Q_0}{2^{|n|}}, n \in \mathbf{Z}, u \in [0,c], v \in [0,c]; \end{array}$$

$$\textbf{(A5):}\quad \begin{array}{l} f\left(n,u,\frac{v}{(|n|+1)^2}\right) \ge \frac{W_0}{2^{|n|}}, n \in [k_1,k_2], u \in [e_2,\frac{e_2}{\mu}], v \in [0,c], \\ g\left(n,u,\frac{v}{(|n|+1)^2}\right) \ge \frac{W_0}{2^{|n|}}, n \in [k_1,k_2], u \in [e_2,\frac{e_2}{\mu}], v \in [0,c]; \end{array}$$

$$\textbf{(A6):}\quad \begin{array}{l} f\left(n,u,\frac{v}{(|n|+1)^2}\right) \le \frac{E_0}{2^{|n|}}, n \in \mathbb{N}_0, u \in [0,e_1], v \in [0,c], \\ g(n,u,v) \le \frac{E_0}{2^{|n|}}, n \in \mathbf{Z}, u \in [0,e_1], v \in [0,c]. \end{array}$$

Then BVP(6) has at least three positive solutions $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) satisfying

$$\sup_{n\in Z} x_1(n), \sup_{n\in\mathbf{Z}} y_1(n) < e_1,\ \ \min_{n\in[k_1,k_2]} x_2(n),\ \min_{n\in k_1,k_2]} y_2(n) > e_2,$$

and

$$\textit{either}\ \sup_{n\in Z} x_3(n)\ \textit{or}\ \sup_{n\in\mathbf{Z}} y_3(n) > e_1,$$

and

$$\textit{either}\ \min_{n\in[k_1,k_2]} x_3(n)\ \textit{or}\ \min_{n\in[k_1,k_2]} y_3(n) < e_2.$$

Proof. Choose $k_1 = 0, k_2 = 102$, $\Phi(x) = \Psi(x) = x$, $p(n) = q(n) = (|n|+1)^2$ and $\alpha_n = \beta_n = \gamma_n = \delta_n = 0$ for all $n \in \mathbf{Z}$. It follows from Theorem 2 and the details are omitted. $\square$

Remark 1. Consider the following boundary value problem of difference system

$$(47)\qquad \begin{cases} \Delta[p(n)\Phi(\Delta x(n))] + f(n, y(n), \Delta y(n)) = 0, & n \in \mathbf{Z},\\ \Delta[q(n)\Psi(\Delta y(n))] + g(n, x(n), \Delta x(n)) = 0, & n \in \mathbf{Z},\\ \lim\limits_{n\to-\infty} \frac{x(n)}{1+\sum\limits_{s=n}^{0}\frac{1}{\Phi^{-1}(p(s))}} - \sum\limits_{n=-\infty}^{+\infty} \alpha_n x(n) = 0, \\ \lim\limits_{n\to-\infty} \frac{y(n)}{1+\sum\limits_{s=n}^{0}\frac{1}{\Psi^{-1}(q(s))}} - \sum\limits_{n=-\infty}^{+\infty} \gamma_n y(n) = 0, \\ \lim\limits_{n\to+\infty} \frac{x(n)}{1+\sum\limits_{s=0}^{n}\frac{1}{\Phi^{-1}(p(s))}} - \sum\limits_{n=-\infty}^{+\infty} \beta_n x(n) = 0, \\ \lim\limits_{n\to+\infty} \frac{y(n)}{1+\sum\limits_{s=0}^{n}\frac{1}{\Psi^{-1}(q(s))}} - \sum\limits_{n=-\infty}^{+\infty} \delta_n y(n) = 0 \end{cases}$$

where (a), (d) and (e) in Section 1 hold and

(b″) $p(n), q(n) > 0$ for all $n \in \mathbf{Z}$ satisfying

$$\sum_{s=0}^{+\infty} \frac{1}{\Phi^{-1}(p(s))} = \sum_{s=-\infty}^{0} \frac{1}{\Phi^{-1}(p(s))} = \sum_{s=0}^{+\infty} \frac{1}{\Psi^{-1}(q(s))} = \sum_{s=-\infty}^{0} \frac{1}{\Psi^{-1}(q(s))} = +\infty.$$

How to establish the existence results on solutions of BVP(47)? we encourage readers to do it.

Remark 2. Consider the following difference system

$$(48)\qquad \begin{cases} \Delta[p(n)\Phi(\Delta x(n))] + f(n, y(n), \Delta y(n)) = 0, & n \in \mathbf{Z},\\ \\ \Delta[q(n)\Psi(\Delta y(n))] + g(n, x(n), \Delta x(n)) = 0, & n \in \mathbf{Z}, \end{cases}$$

where (a) and (e) in Section 1 hold. What conditions guarantee the existence of solutions of (48)? It is interesting to study the solvability of (48).

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Xiaohui Yang
Department of Computers
Guangdong Police College
Guangzhou 510230, P. R. China
E-mail: xiaohui_yang@sohu.com

Yuji Liu
Department of Mathematics and Statics
Guangdong University of Finance andEconomics
Guangzhou 510320, P. R.China
E-mail: yuji_liu@sohu.com