Korean J. Math. **24** (2016), No. 3, pp. 307–317
http://dx.doi.org/10.11568/kjm.2016.24.3.307

ON NI AND QUASI-NI RINGS

Dong Hwa Kim, Seung Ick Lee, Yang Lee, and Sang Jo Yun*

Abstract. Let R be a ring. It is well-known that R is *NI* if and only if $\sum_{i=0}^{n} Ra_iR$ is a nil ideal of R whenever a polynomial $\sum_{i=0}^{n} a_ix^i$ is nilpotent, where x is an indeterminate over R. We consider a condition which is similar to the preceding one: $\sum_{i=0}^{n} Ra_iR$ contains a nonzero nil ideal of R whenever $\sum_{i=0}^{n} a_ix^i$ over R is nilpotent. A ring will be said to be *quasi-NI* if it satisfies this condition. The structure of quasi-NI rings is observed, and various examples are given to situations which raised naturally in the process.

1. Quasi-NI rings

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. We use $N(R)$, $N_*(R)$, and $N^*(R)$ to denote the set of all nilpotent elements, the lower nilradical (i.e., the intersection of all prime ideals), and the upper nilradical (i.e., the sum of all nil ideals) of R, respectively. Note $N^*(R) = \{a \in R \mid RaR \text{ is a nil ideal of } R\}$. The Jacobson radical of R is written by $J(R)$. It is well-known that $N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. The n by n full (resp. upper triangular) matrix ring over R is denoted by $Mat_n(R)$ (resp. $U_n(R)$), and E_{ij} denotes the n by n matrix with 1 (i,j)-entry and zeros elsewhere. $D_n(R)$ and $N_n(R)$ mean the subrings $\{(a_{ij}) \in U_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and $\{(a_{ij}) \in U_n(R) \mid a_{ii} = 0 \text{ for all } i\}$

Received March 23, 2016. Revised July 8, 2016. Accepted July 11, 2016.
2010 Mathematics Subject Classification: 16D25, 16N40, 16S36.
Key words and phrases: quasi-NI ring, NI ring, polynomial ring, matrix ring.
This work was supported by 2-year Research Grant of Pusan National University.
*Corresponding Author.
© The Kangwon-Kyungki Mathematical Society, 2016.
This is an Open Access article distributed under the terms of the Creative commons Attribution Non-Commercial License (http://creativecommons.org/licenses/by-nc/3.0/) which permits unrestricted non-commercial use, distribution and reproduction in any medium, provided the original work is properly cited.

of $U_n(R)$, respectively. We use X to denote a nonempty set (possibly infinite) of commuting indeterminates over given a ring R, and $R[X]$ denotes the polynomial ring with X over R. When $X = \{x\}$ we write $R[x]$ in place of $R[\{x\}]$. $\mathbb{Z}$ denotes the ring of integers and $\mathbb{Z}_n$ denotes the ring of integers modulo n.

A ring R is usually called *reduced* if it has no nonzero nilpotent elements (i.e., $N(R) = 0$). Any reduced ring R satisfies, by help of [11, Proposition 1], that $r_{\sigma(1)}r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$ when $r_1 r_2 \cdots r_n = 0$ for any positive integer n and $r_i \in R$. We will use this fact freely. A ring is usually called *Abelian* if each idempotent is central. Reduced rings are shown to be Abelian by a simple computation.

Marks [12] called a ring R *NI* if $N^*(R) = N(R)$. By the definition we have that a ring R is NI if and only if $N(R)$ forms an ideal of R if and only if $R/N^*(R)$ is reduced. Let $U = U_n(R)$ over a ring R. Then $N(U) = \{m = (m_{ij}) \in U \mid m_{ii} \in N(R) \text{ for all } i\}$ and $N^*(U) = \{m = (m_{ij}) \in U \mid m_{ii} \in N^*(R) \text{ for all } i\}$. So $U/N^*(U) \cong \oplus_{i=1}^{n} R_i$, where $R_i = R/N^*(R)$ for all i. This implies that R is NI if and only if so is U [8, Proposition 4.1(1)]. It is obvious that the class of NI rings contains commutative rings and reduced rings. There exist many non-reduced commutative rings (e.g., $\mathbb{Z}_{n^k}$ for $n, k \geq 2$), and many noncommutative reduced rings (e.g., direct products of noncommutative domains). It is obvious that the Köthe's conjecture (i.e., $N^*(R)$ contains every nil left ideal of R) holds for NI rings.

A ring is called *nil-semisimple* if it has no nonzero nil ideals, following Kim et al. [10]. Nil-semisimple rings are clearly semiprime, but they need not be prime as can be seen by direct products of reduced rings. (Semi)prime rings need not be nil-semisimple by [8, Example 1.2 and Proposition 1.3]. Following Rowen [13, Definition 2.6.5], an ideal P of a ring R is called *strongly prime* if P is prime and R/P is nil-semisimple. While, Handelman and Lawrence [4] used *strongly prime* for rings in which every nonzero ideal contains a finite set whose right annihilator is zero. In this note we follow Rowen's definition.

Let R be a ring. Rowen showed that $N^*(R)$ is the intersection of all strongly prime ideals of R, and $N^*(R)$ is the unique maximal nil ideal of R, in [13, Propositions 2.6.2 and 2.6.7]. Any strongly prime ideal contains a minimal strongly prime ideal by [7, Corollary 2.7]. So we get also that $N^*(R)$ is the intersection of all minimal strongly prime ideals of R. A prime ideal is called *completely prime* if the corresponding prime factor ring is a domain. Hong and Kwak [5, Corollary 13] proved that a ring R is NI if and only if every minimal strongly prime ideal of R is completely prime. It is easily checked that the class of strongly prime ideals contains both completely prime ideals and one-sided primitive ideals.

The following is a simple extension of [8, Lemma 2.1] and [5, Corollary 13].

LEMMA 1.1. *For a ring R the following conditions are equivalent:*

(1) *R is NI;*

(2) *Every subring (possibly without identity) of R is NI;*

(3) *Every minimal strongly prime ideal of R is completely prime;*

(4) *$R/N^*(R)$ is a subdirect product of domains;*

(5) *$R/N^*(R)$ is a reduced ring;*

(6) *$\sum_{i=0}^n Ra_iR$ is nil whenever $\sum_{i=0}^n a_iX_i \in R[X]$ is nilpotent, where every X_i is a finite product of indeterminates in X;*

(7) *$\sum_{i=0}^n Ra_iR$ is nil whenever $\sum_{i=0}^n a_ix^i \in R[x]$ is nilpotent.*

(8) *RaR is nil for any $a \in N(R)$.*

Proof. The equivalences of the conditions (1), (2), (3), (4), and (5) are obtained from [5, Corollary 13] and [8, Lemma 2.1]. (6) $\Rightarrow$ (7) and (7) $\Rightarrow$ (8) are obvious.

(8) $\Rightarrow$ (1): Suppose that the condition holds. Let $a \in N(R)$. Then RaR is nil by the condition, and so $a \in N^*(R)$. This implies $N^*(R) = N(R)$.

(1) $\Rightarrow$ (6): Let R be NI. Then we have $N(R[X]) \subseteq N^*(R)[X]$ from the fact that

$$R[X]/N^*(R)[X] \cong (R/N^*(R))[X]$$

is a reduced ring by (5). Thus if $\sum_{i=0}^n a_iX_i \in R[X]$ is nilpotent, then $a_i \in N^*(R)$ for all i and hence $\sum_{i=0}^n Ra_iR$ is nil. □

Based on the condition (7) in Lemma 1.1, we consider next the following.

DEFINITION 1.2. A ring R is said to be *quasi-NI* provided that $\sum_{i=0}^n Ra_iR$ contains a nonzero nil ideal of R whenever a nonzero polynomial $\sum_{i=0}^n a_ix^i$ over R is nilpotent.

The following is shown easily, but useful in our process.

Lemma 1.3. *For a ring R the following conditions are equivalent:*

(1) *R is quasi-NI;*

(2) *RaR contains a nonzero nil ideal of R for any $0 \neq a \in N(R)$;*

(3) *$\sum_{i=0}^{n} Ra_iR$ contains a nonzero nil ideal of R whenever $\sum_{i=0}^{n} a_iX_i \in R[X]$ is nilpotent, where every X_i is a finite product of indeterminates in X.*

Proof. (2) $\Rightarrow$ (1). Suppose $0 \neq \sum_{i=0}^{n} a_ix^i \in N(R[x])$. Let $0 \leq m \leq 0$ be the smallest integer such that $a_m \neq 0$. Then $a_m \in N(R)$ clearly. So, by the condition, Ra_mR contains a nonzero nil ideal of R, I say, entailing that $\sum_{i=0}^{n} Ra_iR$ contains I.

(1) $\Rightarrow$ (2) and (3) $\Rightarrow$ (1) are obvious. Let $0 \neq f(X) = \sum_{i=0}^{n} a_iX_i \in R[X]$ be nilpotent in $R[X]$ to prove (2) $\Rightarrow$ (3). Then the number of indeterminates occur in the polynomial $f(X)$, $\{x_1, \ldots, x_k\}$ say. So we consider $f(X)$ as a polynomial in $R[x_1, \ldots, x_n]$. We can write

$$f(X) = g_1(x_1, \ldots, x_{n-1})x_n^{h_1} + \cdots + g_s(x_1, \ldots, x_{n-1})x_n^{h_s} \in R[x_1, \ldots, x_{n-1}][x_n],$$

where $g_l(x_1, \ldots, x_{n-1}) \in R[x_1, \ldots, x_{n-1}]$, $h_1 < \cdots < h_s$, and $g_1(x_1, \ldots, x_{n-1}) \neq 0$. Since $f(X)$ is nilpotent, $g_1(x_1, \ldots, x_{n-1})$ is also nilpotent. Here if $g_1(x_1, \ldots, x_{n-1}) \in R$ then $g_1(x_1, \ldots, x_{n-1}) = a_\alpha$ for some α. Otherwise, we write

$g_1(x_1, \ldots, x_{n-1}) = k_1(x_1, \ldots, x_{n-2})x_{n-1}^{t_1} + \cdots + k_u(x_1, \ldots, x_{n-2})x_{n-1}^{t_v} \in R[x_1, \ldots, x_{n-2}][x_{n-1}]$,

where $k_w(x_1, \ldots, x_{n-2}) \in R[x_1, \ldots, x_{n-2}]$, $t_1 < \cdots < t_v$, and $k_1(x_1, \ldots, x_{n-2}) \neq 0$. Since the polynomial $g_1(x_1, \ldots, x_{n-1})$ is nilpotent, $k_1(x_1, \ldots, x_{n-2})$ is also nilpotent. Proceeding in this method, we can get finally a nilpotent polynomial

$$b_0x_\gamma^{y_0} + b_1c_1(x_1, \ldots, x_{\gamma-1})x_\gamma^{y_1} + \cdots + b_zc_p(x_1, \ldots, x_{\gamma-1})x_\gamma^{y_d}$$

in $R[x_1, \ldots, x_{\gamma-1}][x_\gamma]$, where $\gamma \geq 1$, $b_0 \neq 0$ and $y_0 < y_1 < \cdots < y_d$. Note that $b_0 \in N(R)$ and $b_0 = a_\beta$ for some β.

Now if R satisfies the condition (2) then Rab_0R contains a nonzero nil ideal of R, I say. So $\sum_{i=0}^{n} Ra_iR$ contains I. □

The following is an immediate consequence of the preceding lemma.

Corollary 1.4. *If a ring R is quasi-NI, then we have the following.*

(1) *$N(R) \neq 0$ implies $N^*(R) \neq 0$.*

(2) *$N^*(R) = 0$ implies $N(R) = 0$.*

Proof. Assume $N(R) \neq 0$, and take $0 \neq a \in N(R)$. If R is quasi-NI then RaR contains a nonzero nil ideal of R by Lemma 1.3, entailing $N^*(R) \neq 0$. (1) and (2) are contrapositions each other. $\square$

NI rings are quasi-NI by Lemma 1.1, but the converse need not hold as we see in the following. $\prod$ is used to express a direct product. Recall that an element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. The *left regular* can be defined similarly. An element is *regular* if it is both left and right regular (i.e., not a zero divisor).

PROPOSITION 1.5. (1) *Let R be a ring with $N^*(R) \neq 0$ and $S = Mat_n(R)$. Suppose that $N^*(R)$ is nilpotent and every element in $R \backslash N^*(R)$ is regular in R. Then SAS contains a nonzero nilpotent ideal of S for all $0 \neq A \in Mat_n(R)$, and especially $Mat_n(R)$ is quasi-NI.*

(2) *$Mat_n(R)$ is not quasi-NI over any simple ring R when $n \geq 2$.*

(3) *$Mat_n(R)$ is not quasi-NI over any domain R when $n \geq 2$.*

(4) *$Mat_n(R)$ is not NI over any ring R when $n \geq 2$.*

(5) *Let R_i be a quasi-NI ring for each $i \in I$. Then $R = \prod_{i \in I} R_i$ is quasi-NI.*

Proof. (1) Since $N^*(R)$ is nilpotent, $Mat_n(N^*(R))$ is also nilpotent. This implies $Mat_n(N^*(R)) \subseteq N^*(S)$. We will show $Mat_n(N^*(R)) = N^*(S)$. Note that $N^*(R) = N(R)$ since every element in $R \backslash N^*(R)$ is regular by hypothesis.

Consider $Mat_n(R)/Mat_n(N^*(R))$. Note $Mat_n(R)/Mat_n(N^*(R)) \cong Mat_n(R/N^*(R))$. Let $B = (b_{ij}) \in N^*(S)$. Then $b_{ij}E_{11} = E_{1i}BE_{j1} \in N^*(S)$ for all i and j, and so $b_{ij} \in N(R)$. So $b_{ij} \in N^*(R)$ because $N(R) = N^*(R)$, entailing $B \in Mat_n(N^*(R))$. Consequently $Mat_n(N^*(R)) = N^*(S)$. We then obtain $N^*(S) \neq 0$ from $N^*(R) \neq 0$.

Let $0 \neq A = (a_{ij}) \in S$. If $A \in N^*(S)$ then SAS is clearly a nonzero nilpotent ideal of S.

Assume $A \notin N^*(S)$. We claim that SAS contains a nonzero nilpotent ideal of S.

If some nonzero entry of A, say a_{ij}, is contained in $N^*(R)$, then $SE_{1i}AE_{j1}S$ is a nonzero nilpotent ideal of S because $SE_{1i}AE_{j1}S \subseteq Mat_n(N^*(R)) = N^*(S)$. Note $SE_{1i}AE_{j1}S \subseteq SAS$.

If every nonzero entry of A is contained in $R \backslash N^*(R)$, then $S(bA)S$ is a nonzero nilpotent ideal of S for all $0 \neq b \in N^*(R)$ because $0 \neq bA \in Mat_n(N^*(R))$ (since every nonzero entry of A is regular) and $S(bA)S \subseteq Mat_n(N^*(R)) = N^*(S)$, where $bA = (ba_{ij})$. Note $S(bA)S \subseteq SAS$.

It is an immediate consequence that S is quasi-NI.

(2) Let R be a simple ring. Then $Mat_n(R)E_{12}Mat_n(R) = Mat_n(R)$ for $E_{12} \in N(Mat_n(R))$. But $N^*(Mat_n(R)) = 0$ and so $Mat_n(R)$ is not quasi-NI.

(3) Let R be a domain and consider $S = Mat_n(R)$ for $n \geq 2$. Assume $N^*(S) \neq 0$ and take $A = (a_{ij}) \neq 0$ in $N^*(S)$. Then some nonzero entry of A, say a_{ij}, is regular; hence $SE_{1i}AE_{j1}S$ is a non-nil ideal of S by the existence of the non-nilpotent matrix $a_{ij}E_{11}$ contained in $SE_{1i}AE_{j1}S$. This contradicts $SE_{1i}AE_{j1}S \subseteq SAS \subseteq N^*(S)$. Thus $N^*(S) = 0$, and this implies that $SE_{12}S$ does not contain a nonzero nil ideal of S. So $Mat_n(R)$ is not quasi-NI.

(4) Let R be a ring and consider $Mat_n(R)$ for $n \geq 2$. E_{12} and E_{21} are nilpotent but $E_{12} + E_{21} \notin N(S)$, concluding that $Mat_n(R)$ is not NI.

(5) Let $0 \neq f(x) = \sum_{k=1}^{m} b_k x^k \in N(R[x])$. Then there exists a nonzero nilpotent coefficient of $f(x)$, say b_t. Let $b_t = a = (a_i)_{i \in I}$ with $a_s \neq 0$. Note $a_i \in N(R_i)$. Let $e_i \in R$ be such that $e_i(i) = 1_{R_i}$ and $e_i(j) = 0_{R_j}$ for all $j \neq i$. Since R_s is quasi-NI, $R_s a_s R_s$ contains a nonzero nil ideal of R_s, say N, and moreover RaR ($\supseteq Re_s aR$) contains a nonzero nil ideal M of R such that $e_s(M) = N$ and $e_i(M) = 0$ for all $i \neq s$. Thus $\sum_{k=1}^{m} Rb_kR$ contains the nonzero nil ideal M of R. □

Any local ring R with nonzero nil Jacobson radical (e.g., $D_n(R)$ for $n \geq 2$ over a division ring R) satisfies the condition in Proposition 1.5(1), so $Mat_n(R)$ is quasi-NI. The condition that every element in $R \backslash N^*(R)$ is regular in Proposition 1.5(1) is not superfluous by the following.

Example 1.6. Let $R = \mathbb{Z} \oplus \mathbb{Z}_4$. Note that R is a commutative (hence NI) ring with $N^*(R) = 0 \oplus 2\mathbb{Z}_4$, and that $(1,0) \in R \backslash N^*(R)$ is not regular. Let $S = Mat_2(R)$ and consider $M = \begin{pmatrix} (0,0) & (1,0) \\ (0,0) & (0,0) \end{pmatrix} \in N(S)$. In fact, $M^2 = 0$ and

$$SMS = \begin{pmatrix} \mathbb{Z} \oplus 0 & \mathbb{Z} \oplus 0 \\ \mathbb{Z} \oplus 0 & \mathbb{Z} \oplus 0 \end{pmatrix}.$$

But we have

$$SMS = Mat_2(\mathbb{Z} \oplus 0) \cong Mat_2(\mathbb{Z}).$$

Assume here that S is quasi-NI. Then SMS contains a nonzero nil ideal of S. However this is impossible because $Mat_2(\mathbb{Z})$ is not quasi-NI by Proposition 1.5(3). Thus S is not quasi-NI.

In the following we see another kind of quasi-NI ring that is not NI.

EXAMPLE 1.7. We use the ring in [6, Examples 1.6]. Let K be a field and define $D_n = K\{x_n\}$, a free algebra generated by x_n, with a relation $x_n^{n+2} = 0$ for each nonnegative integer n. Then clearly $D_n \cong K[x]/(x^{n+2})$, where (x^{n+2}) is the ideal of $K[x]$ generated by x^{n+2}. Next let $R_n = \begin{pmatrix} D_n & x_nD_n \\ x_nD_n & D_n \end{pmatrix}$ be a subring of $Mat_2(D_n)$. Then $N^*(R_n) = \begin{pmatrix} x_nD_n & x_nD_n \\ x_nD_n & x_nD_n \end{pmatrix}$. So we get $R_n/N^*(R_n) \cong K \oplus K$, entailing that R_n is NI.

Set next $R = \prod_{n=0}^{\infty} R_n$. Then R is quasi-NI by Lemma 1.5(4) because every R_n is NI. However R is not NI by [8, Example 2.5].

The class of NI rings is closed under subrings by [8, Proposition 2.4(2)]. But this result is not valid for quasi-NI rings.

EXAMPLE 1.8. Let F be a division ring and $R = D_k(F)$ for $k \geq 2$. Consider next $Mat_n(R)$ for $n \geq 2$. Then

$$N^*(R) = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\} = N(R),$$

noting that $N^*(R) \neq 0$ and $N^*(R)^k = 0$. Moreover $R \backslash N^*(R)$ is correspondent to $\mathbb{Z}\backslash\{0\}$, so every element in $R\backslash N^*(R)$ is regular in R. Thus $Mat_n(R)$ is quasi-NI by Proposition 1.5(1).

Consider next the subring $Mat_n(F)$ for $Mat_n(R)$, noting that F is a subring of R. However $Mat_n(F)$ is not quasi-NI by Proposition 1.5(2) or Proposition 1.5(3).

Considering Corollary 1.4, one may ask whether $N(R) \neq 0$ implies $N_*(R) \neq 0$ for a quasi-NI ring R. But the answer is negative by the following.

EXAMPLE 1.9. We follow the construction of [8, Example 1.2]. Let S be a reduced ring and $U_n = U_{2^n}(S)$ for all $n \geq 1$. Define a map $\sigma : U_n \to U_{n+1}$ by $B \mapsto \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$. Then U_n can be considered as a subring of U_{n+1} via σ (i.e., $B = \sigma(B)$ for $B \in U_n$). Set $R = \cup_{n=1}^{\infty} U_n$. Then R is semiprime by [9, Theorem 2.2(1)]. But

$$N^*(R) = \{B \in R \mid \text{ all the diagonal entries of } B \text{ are zero}\} = N(R).$$

So $R/N^*(R)$ is a reduced ring, and so R is NI. So $N_(R) \neq 0$ but $N_*(R) = 0$.

2. About ordinary ring extensions

In this section we investigate several kinds of ring extensions of quasi-NI rings which can be helpful to related studies.

PROPOSITION 2.1. (1) $U_n(R)$ *is quasi-NI for any ring* R *when* $n \geq 2$.
(2) $D_n(R)$ *is quasi-NI for any ring* R *when* $n \geq 2$.

Proof. (1) Let $T = U_n(R)$ and $0 \neq A = (a_{ij}) \in T$. If $A \in N_n(R)$ then $0 \neq TAT \subseteq N_n(R)$.

Assume $A \notin N_n(R)$. Then $a_{kk} \neq 0$ for some k. So

$$0 \neq T(AE_{k(k+1)})T \subseteq TAT \text{ and } T(AE_{k(k+1)})T \subseteq N_n(R).$$

Thus TAT contains a nonzero nil ideal of T, and so T is quasi-NI.

The proof for $D_n(R)$ is similar. □

This result can be compared with the facts (1), (2), (3), and (4) in Proposition 1.5. One can also compare this with the fact that a ring R is NI if and only if $U_n(R)$ is NI (if and only if $D_n(R)$ is NI) [8, Proposition 4.1(1)].

We use $\oplus$ to denote the direct sum. Let R be an algebra (with or without identity) over a commutative ring S. Due to Dorroh [2], the *Dorroh extension* of R by S is the Abelian group $R \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in R$ and $s_i \in S$.

PROPOSITION 2.2. *Let* R *be an algebra (with identity) over a commutative reduced ring* S. *If* R *is quasi-NI then the Dorroh extension* D *of* R *by* S *is quasi-NI.*

Proof. Note first that $s \in S$ is identified with $s1 \in R$ because R has the identity, and so we have $R = \{r + s \mid (r, s) \in D\}$.

Let $0 \neq (a, s) \in N(D)$. Then $s = 0$ since S is a reduced ring. This implies $0 \neq a \in N(R)$. Since R is quasi-NI, RaR contains a nonzero nil ideal of R, I say. Consider

$$J = I \oplus 0 = \{(r, s) \mid r \in I \text{ and } s = 0\}.$$

For all $(u, v) \in D$ and $(r, 0) \in J$,

$$(u, v)(r, 0) = ((u + v)r, 0) \in J \text{ and } (r, 0)(u, v) = (r(u + v), 0) \in J$$

because $u + v \in R$ by the argument above and I is an ideal of R. Moreover J is nil because I is nil. Thus $D(a, s)D$ contains the nonzero nil ideal J of D because

$$J = I \oplus 0 \subseteq RaR \oplus 0 = D(a, 0)D,$$

noting $R = \{r + s \mid (r, s) \in D\}$ and

$$\sum_{\text{finite}} (r, s)(a, 0)(r', s') = \sum_{\text{finite}} (r + s)a(r' + s'),$$

where $(r, s), (r', s') \in D$. Therefore D is also a quasi-NI ring. □

Following Goodearl [3], a ring R is called *von Neumann regular* (simply, *regular*) if for every $a \in R$ there exists $b \in R$ such that $aba = a$. Every regular ring R is clearly semiprimitive (i.e., $J(R) = 0$) because ab is a nonzero idempotent for all $0 \neq a \in R$. So we have the following equivalence for regular rings.

PROPOSITION 2.3. *For a regular ring R the following conditions are equivalent:*
(1) *R is quasi-NI;*
(2) *R is NI;*
(3) *R is Abelian;*
(4) *R is reduced.*

Proof. We have first $N^*(R) = 0$ for a regular ring R. So if R is quasi-NI then $N(R) = 0$ (i.e., R is reduced) by Lemma 1.3. Reduced rings are clearly NI both and Abelian. Abelian regular rings are reduced by [3, Theorem 3.2]. □

Following the literature, a ring R is called *π-regular* if for each $a \in R$ there exist a positive integer $n = n(a)$, depending on a, and $b \in R$ such that $a^n = a^n b a^n$. Regular rings are obviously π-regular, letting $n(a) = 1$ for all a. So it is natural to ask whether a π-regular ring R is reduced when R is quasi-NI. However the answer is negative by the following.

EXAMPLE 2.4. Let A be a division ring and $R = U_n(A)$ or $R = D_n(A)$ for $n \geq 2$. Then R is π-regular by [1, Corollary 6], and R is not regular by the existence of nonzero $N^*(R)$. Moreover R is quasi-NI by Proposition 2.1, but R is not reduced.

We do not know the answer of the following:
Question. Does the Köthe's conjecture hold for quasi-NI rings?

References

[1] G.F. Birkenmeier, J.Y. Kim and J.K. Park, *A connection between weak regularity and the simplicity of prime factor rings*, Proc. Amer. Math. Soc. **122** (1994), 53–58.

[2] J.L. Dorroh, *Concerning adjunctins to algebras*, Bull. Amer. Math. Soc. **38** (1932), 85–88.

[3] K.R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, 1979.

[4] D. Handelman and J. Lawrence, *Strongly prime rings*, Tran. Amer. Math. Soc. **211** (1975), 209–223.

[5] C.Y. Hong and T.K. Kwak, *On minimal strongly prime ideals*, Comm. Algebra **28** (2000), 4867–4878.

[6] C. Huh, H.K. Kim and Y. Lee, *On rings whose strongly prime ideals are completely prime*, Comm. Algebra **26** (1998), 595–600.

[7] C. Huh, C.I. Lee and Y. Lee, *On rings whose strongly prime ideals aAre completely prime*, Algebra Colloq. **17** (2010), 283–294.

[8] S.U. Hwang, Y.C. Jeon and Y. Lee, *Structure and topological conditions of NI rings*, J. Algebra **302** (2006), 186–199.

[9] Y.C. Jeon, H.K. Kim, Y. Lee and J.S. Yoon, *On weak Armendariz rings*, Bull. Korean Math. Soc. **46** (2009), 135–146.

[10] N.K. Kim, Y. Lee and S.J. Ryu, *An ascending chain condition on Wedderburn radicals*, Comm. Algebra **34** (2006), 37–50.

[11] J. Lambek, *On the representation of modules by sheaves of factor modules*, Canad. Math. Bull. **14** (1971), 359–368.

[12] G. Marks, *On 2-primal Ore extensions*, Comm. Algebra **29** (2001), 2113–2123.

[13] L.H. Rowen, Ring Theory, *Academic Press*, San Diego (1991).

Dong Hwa Kim
Department of Mathematics Education
Pusan National University
Busan 46241, Korea
E-mail: dhgim@pusan.ac.kr

Seung Ick Lee
Department of Mathematics
Pusan National University
Busan 46241, Korea
E-mail: maick@hanmail.net

Yang Lee
Department of Mathematics
Pusan National University
Busan 46241, Korea
E-mail: ylee@pusan.ac.kr

Sang Jo Yun
Department of Mathematics
Pusan National University
Busan 46241, Korea
E-mail: pitt0202@hanmail.net