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A GENERIC RESEARCH ON NONLINEAR NON-CONVOLUTION TYPE SINGULAR INTEGRAL OPERATORS

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Abstract. In this paper, we present some general results on the pointwise convergence of the non-convolution type nonlinear singular integral operators in the following form:

$$T_\lambda(f;x)=\int\limits_{\Omega}K_\lambda\left(t,x,f\left(t\right)\right)dt,\ x\in\Psi,\ \lambda\in\Lambda,$$

where $\Psi=\langle a,b\rangle$ and $\Omega=\langle A,B\rangle$ stand for arbitrary closed, semi-closed or open bounded intervals in $\mathbb{R}$ or these set notations denote $\mathbb{R}$, and Λ is a set of non-negative numbers, to the function $f\in L_{p,w}\left(\Omega\right)$, where $L_{p,w}\left(\Omega\right)$ denotes the space of all measurable functions f for which $\left|\frac{f}{w}\right|^p$ $(1\leq p<\infty)$ is integrable on Ω, and $w:\mathbb{R}\to\mathbb{R}^+$ is a weight function satisfying some conditions.

1. Introduction

Approximating functions with simpler functions is one of the major problems of mathematical analysis. The main aim is to make the approximation error smaller, i.e., to make simpler functions closer to the original function. Here, simpler functions refer to the family or sequence

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of functions equipped with good properties, such as continuity, differentiability and integrability. There are many approximation methods in the literature. In particular, approximation of integrable functions by a family of integrable functions is a classical method used by many researchers. To make this sentence clear, we outline some of the significant studies as follows:

Taberski [27] analyzed the pointwise convergence of integrable functions and the approximation properties of derivatives of integrable functions in $L_1 \langle -\pi, \pi \rangle$, where $\langle -\pi, \pi \rangle$ is an arbitrary closed, semi-closed or open interval, by the following integral operator

$$L_\lambda (f;x) = \int_{-\pi}^{\pi} f(t) K_\lambda (t-x)\, dt, \quad x \in \langle -\pi, \pi \rangle, \quad \lambda \in \Lambda \subset \mathbb{R}_0^+, \tag{1.1}$$

where $K_\lambda (t)$ is the kernel fulfilling appropriate conditions and Λ is a given set of non-negative numbers with accumulation point λ_0. Further, the singularity condition is satisfied by $K_\lambda (t)$, i.e., $K_\lambda (0)$ tends to infinity as λ tends to λ_0.

Following Taberski [27], Gadjiev [12] and Rydzewska [26] studied pointwise convergence of the operators of type (1.1) at different characteristic points of integrable functions. In addition, Bardaro [2] studied the rate of convergence of the linear singular integral operators in different function spaces. Further, in [3], Bardaro and Vinti obtained some approximation properties of certain non-convolution type integral operators. Later on, Alexits [1], Mamedov [15] and Esen [10] presented necessary conditions satisfied by kernel functions in order to obtain a good approximation, separately.

In the year 1983, Musielak [21] studied the convergence of convolution type nonlinear integral operators in the following form:

$$S_w(f;y) = \int_G K_w(x-y, f(x))dx, \quad y \in G, \quad w \in \Lambda, \tag{1.2}$$

where G is a locally compact Abelian group equipped with Haar measure and $\Lambda \neq \emptyset$ is an index set with any topology, and he extended the scope of the singularity assumption via replacing the linearity property of the integral operators of type (1.1) by an assumption of Lipschitz condition for K_w with respect to second variable. In [22], Musielak stepped up his previous analysis by presenting the significant approximation results for the operators of type (1.2) in generalized Orlicz spaces. After these pioneering studies, Swiderski and Wachnicki [23] investigated the pointwise convergence of the operators of type (1.2) at $p-$Lebesgue points of functions $f \in L_p(-\pi,\pi)$, where $1 \leq p < \infty$. The convergence properties of specific types of nonlinear integral operators were investigated, for example, the pointwise convergence of nonlinear Mellin type convolution operators at Lebesgue points was studied by Bardaro and Mantellini [6]. In [14], Karsli investigated the approximation properties of non-convolution type nonlinear integral operators and established the rate of pointwise convergence. Some further approximation results concerning approximation by non-convolution type singular integral operators can be found in [10, 11]. Based on the results given by Esen [10], Guller *et al.* [13] presented some approximation theorems. For some advanced results concerning the convergence of several types of nonlinear singular integral operators in modular function spaces, we refer the reader to the monograph by Bardaro *et al.* [5] and the articles by Bardaro and Vinti [3, 4]. Recently, some weighted pointwise approximation results for the nonlinear counterpart of the operator of type (1.1) were presented in [28] using the approximation method presented by Esen [10]. For some studies concerning the applications of approximation of functions by linear and nonlinear operators, we refer the reader to see the works [8, 9, 16-20, 24].

In this paper, we prove the convergence of non-convolution type nonlinear singular integral operators designated by

$$T_{\lambda}(f;x)=\int\limits_{\Omega}K_{\lambda}\left(t,x,f\left(t\right)\right)dt,\quad x\in\Psi,\quad \lambda\in\Lambda, \tag{1.3}$$

where $\Psi=\langle a,b\rangle$ and $\Omega=\langle A,B\rangle$ stand for arbitrary closed, semi-closed or open bounded intervals in $\mathbb{R}$ or the indicated set notations represent $\mathbb{R}$, and Λ is a set of non-negative numbers, to the function $f\in L_{p,w}\left(\Omega\right)$, where $L_{p,w}\left(\Omega\right)$ is the space of all measurable functions f for which $\left|\frac{f}{w}\right|^{p}$ $(1\leq p<\infty)$ is integrable on Ω, and $w:\mathbb{R}\to\mathbb{R}^{+}$ is a weight function, at a common μ-generalized Lebesgue point of the functions $\frac{f}{w}$ and w. In this work, the weighted approximation method presented by Esen [10] is harnessed.

The paper is organized as follows: In Section 2, we introduce basic definitions. In Section 3, we prove the existence of the operators type

(1.3). In Section 4, we present two theorems concerning the pointwise convergence of $T_\lambda(f;x)$ to the value $f(x_0)$ as (x,λ) tends to (x_0,λ_0), where x_0 is a $\mu-$generalized Lebesgue point of $f \in L_p(\Omega)$. In Section 5, we present two theorems concerning the pointwise convergence of $T_\lambda(f;x)$ to the value $f(x_0)$ as (x,λ) tends to (x_0,λ_0), where x_0 is a common $\mu-$generalized Lebesgue point of $f \in L_{p,w}(\Omega)$ and $w \in L_p(\Omega)$. In Section 6, we give two theorems concerning the rate of pointwise convergence.

2. Preliminaries

Definition 2.1. A point $x_0 \in \Omega$ is called a $\mu-$generalized Lebesgue point of the function $f \in L_p(\Omega)$ if

$$\lim_{h\to 0}\left(\frac{1}{\mu(h)}\int_0^h |f(t+x_0)-f(x_0)|^p\,dt\right)^{\frac{1}{P}}=0,\quad 1\le p<\infty,$$

where $\mu : \mathbb{R}\to\mathbb{R}$ is an increasing and absolutely continuous function on $[0,B-A]$ with $\mu(0)=0$ [26].

Using the kernel properties given in [11, 14], the following definition is obtained:

Definition 2.2. (*Class A_w*) Let Λ be a set of non-negative set of numbers with accumulation point λ_0. Suppose that for every fixed $x \in \Psi$ there exists a point $x_0 \in \Omega$. We say that the family of functions $(K_\lambda)_{\lambda\in\Lambda}$, $K_\lambda : \mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ belongs to *class A_w* if the following conditions are satisfied:

a. $K_\lambda(t,x,0)=0$, for every $t,x\in\mathbb{R}$ and $\lambda\in\Lambda$.

b. The function $L_\lambda : \mathbb{R}\times\mathbb{R}\to\mathbb{R}_0^+$ which is integrable for every $t,x\in\mathbb{R}$ and for each $\lambda\in\Lambda$ exists such that the inequality

$$|K_\lambda(t,x,u)-K_\lambda(t,x,v)|\le L_\lambda(t,x)\,|u-v|$$

holds for every $t,x,u,v\in\mathbb{R}$ and for each $\lambda\in\Lambda$.

c. $\lim_{(x,\lambda)\to(x_0,\lambda_0)}\left|\int\limits_{\mathbb{R}} K_\lambda(t,x,u)dt-u\right|=0$, for every $u\in\mathbb{R}$.

d. $\lim_{(x,\lambda)\to(x_0,\lambda_0)}\left(\sup_{t\in\mathbb{R}\setminus\langle x_0-\xi,x_0+\xi\rangle} L_\lambda(t,x)\right)=0$, for every $\xi>0$.

e. $\lim_{(x,\lambda)\to(x_0,\lambda_0)}\left(\int_{\mathbb{R}\setminus\langle x_0-\xi,x_0+\xi\rangle} L_\lambda(t,x)dt\right)=0$, for every $\xi>0$.

f. There exists $\delta_0 > 0$ such that $L_\lambda(t,x)$ is non-decreasing as a function of t on $\langle x_0 - \delta_0, x_0]$ and non-increasing as a function of t on $[x_0, x_0 + \delta_0\rangle$, for each fixed $\lambda \in \Lambda$.

g. $\left\| \frac{w(\cdot)}{w(x)} L_\lambda(\cdot, x) \right\|_{L_1(\mathbb{R})} \leq M < \infty$, where M is a constant whose value does not depend on $x \in \Psi$ and $\lambda \in \Lambda$.

h. $\|\alpha L_\lambda\|_{L_q(\mathbb{R}\times\mathbb{R})} \leq N < \infty$, where N is a constant whose value does not depend on $\lambda \in \Lambda$, $1 < q < \infty$ and $\alpha(t,x) = \frac{w(t)}{w(x)}$.

Throughout this paper K_λ belongs to class A_w.

3. Existence of the Operators

THEOREM 3.1. *Let $f \in L_{1,w}(\Omega)$. Then, $T_\lambda \in L_{1,w}(\Omega)$ and the following inequality*

$$\|T_\lambda(f;x)\|_{L_{1,w}(\Omega)} \leq \left\| \frac{w(\cdot)}{w(x)} L_\lambda(\cdot, x) \right\|_{L_1(\mathbb{R})} \|f\|_{L_{1,w}(\Omega)}$$

holds for every $x \in \Psi$ and $\lambda \in \Lambda$.

Proof. Assume that $\Psi = \langle a, b\rangle$ and $\Omega = \langle A, B\rangle$ stand for arbitrary closed, semi-closed or open bounded intervals in $\mathbb{R}$.

Using conditions (a), (b) of class A_w and Fubini's Theorem (see, e.g., [7]), we may write

$$\begin{aligned}
\|T_\lambda(f;x)\|_{L_{1,w}(\Omega)} &= \int\limits_\Omega \frac{1}{w(x)} \left| \int\limits_\Omega K_\lambda(t,x,f(t))dt \right| dx \\
&\leq \int\limits_\Omega \frac{1}{w(x)} \int\limits_\Omega \left| f(t) \frac{w(t)}{w(t)} L_\lambda(t,x) \right| dtdx \\
&\leq \int\limits_\Omega \left| \frac{f(t)}{w(t)} \right| \left[\int\limits_{\mathbb{R}} \frac{w(t)}{w(x)} L_\lambda(t,x) dx \right] dt \\
&\leq M \|f\|_{L_{1,w}(\Omega)}.
\end{aligned}$$

The proof is completed for the indicated case. One may prove the assertion for the case $\Psi = \Omega = \mathbb{R}$ by following similar steps. Thus the proof is completed. □

Theorem 3.2. *Let $f \in L_{p,w}(\Omega)$. Then, $T_\lambda \in L_{q,w}(\Omega)$ and the following inequality*

$$\|T_\lambda(f;x)\|_{L_{q,w}(\Omega)} \le \|\alpha L_\lambda\|_{L_q(\mathbb{R}\times\mathbb{R})} \|f\|_{L_{p,w}(\Omega)}$$

holds for every $x \in \Psi$ and $\lambda \in \Lambda$. Here, $\frac{1}{q} + \frac{1}{p} = 1$ provided that $1 < p, q < \infty$.

Proof. Suppose that $\Psi = \langle a, b\rangle$ and $\Omega = \langle A, B\rangle$ stand for arbitrary closed, semi-closed or open bounded intervals in $\mathbb{R}$.

Let us define a new function by

$$g(t) := \begin{cases} f(t), & if \quad t \in \Omega, \\ 0, & if \quad t \in \mathbb{R}\backslash\Omega. \end{cases}$$

It is easy to see that the following inequality

$$\begin{aligned} \|T_\lambda(f;x)\|_{L_q(\Omega)} &= \|T_\lambda(g;x)\|_{L_q(\Omega)} \\ &= \left(\int_\Omega \frac{1}{w^q(x)} \left|\int_\mathbb{R} K_\lambda(t,x,g(t))dt\right|^q dx\right)^{\frac{1}{q}} \\ &\le \left(\int_\Omega \frac{1}{w^q(x)} \left(\int_\mathbb{R} |g(t)L_\lambda(t,x)|\, dt\right)^q dx\right)^{\frac{1}{q}} \end{aligned}$$

holds. Now, applying Hölder's inequality (see, e.g., [25]) to the above inequality, we obtain

$$\begin{aligned} &\|T_\lambda(f;x)\|_{L_q(\Omega)} \\ &\le \left(\int_\Omega \frac{1}{w^q(x)} \left(\left(\int_\mathbb{R} \left|\frac{g(t)}{w(t)}\right|^p dt\right)^{\frac{1}{p}} \left(\int_\mathbb{R} |w(t)L_\lambda(t,x)|^q\, dt\right)^{\frac{1}{q}}\right)^q dx\right)^{\frac{1}{q}} \\ &\le \|f\|_{L_{p,w}(\Omega)} \left(\int_\Omega \left(\int_\mathbb{R} \left|\frac{w(t)}{w(x)} L_\lambda(t,x)\right|^q dt\right) dx\right)^{\frac{1}{q}} \\ &\le \|f\|_{L_{p,w}(\Omega)} \left(\int_\mathbb{R} \left(\int_\mathbb{R} \left|\frac{w(t)}{w(x)} L_\lambda(t,x)\right|^q dt\right) dx\right)^{\frac{1}{q}} \\ &\le \|f\|_{L_{p,w}(\Omega)} \|\alpha L_\lambda\|_{L_q(\mathbb{R}\times\mathbb{R})} \\ &\le N \|f\|_{L_{p,w}(\Omega)} . \end{aligned}$$

Hence, the proof is completed for the indicated case. One may prove the assertion for the case $\Psi = \Omega = \mathbb{R}$ by following similar steps. Thus the proof is completed. $\square$

4. Convergence at Characteristic Points

Throughout this section we assume that $w(t) = 1$. Also, in the following theorem, $\Psi = \langle a, b\rangle$ and $\Omega = \langle A, B\rangle$ denote arbitrary closed, semi-closed or open bounded intervals in $\mathbb{R}$.

THEOREM 4.1. *Let $x_0 \in \Omega$ be a $\mu-$generalized Lebesgue point of function $f \in L_p(\Omega)$. Then,*

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |T_\lambda(f;x) - f(x_0)| = 0$$

on any set Z on which the function

$$\int_{x_0-\delta}^{x_0+\delta} L_\lambda(t,x) \left|\{\mu(|x_0-t|)\}'_t\right| dt$$

is bounded as (x,λ) tends to (x_0,λ_0).

Proof. We prove the theorem for the case $1 < p < \infty$. The proof of the case $p = 1$ is quite similar to the proof of the case $1 < p < \infty$ and it is omitted.

Let $\langle x_0 - \delta, x_0 + \delta\rangle \subset \Omega$, for fixed $\delta > 0$, where x_0 is a $\mu-$generalized Lebesgue point of function $f \in L_p(\Omega)$. Set $E(x,\lambda) := |T_\lambda(f;x) - f(x_0)|$. According to condition (c), we shall write

$$\begin{aligned} E(x,\lambda) &= \left|\int_{\Omega} K_\lambda(t,x,f(t))dt - f(x_0)\right| \\ &\le \int_{\Omega} |f(t)-f(x_0)|\, L_\lambda(t,x)dt + \left|\int_{\mathbb{R}} K_\lambda(t,x,f(x_0))dt - f(x_0)\right| \\ &\quad + |f(x_0)| \int_{\mathbb{R}\setminus\Omega} L_\lambda(t,x)dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By condition (e), $I_3 \to 0$ as (x,λ) tends to (x_0,λ_0). Using well known inequality $(m+n)^p \leq 2^p(m^p+n^p)$, where m and n are positive real numbers (see, e.g., [25]), we have

$$\begin{aligned}
&(I_1+I_2)^p \\
\leq\ & 2^p(I_1^p+I_2^p) \\
=\ & 2^p\left(\int\limits_{\Omega} |f(t)-f(x_0)|\, L_\lambda(t,x)dt\right)^p + 2^p\left|\int\limits_{\mathbb{R}} K_\lambda(t,x,f(x_0))dt - f(x_0)\right|^p \\
=\ & 2^p(I_4+I_5).
\end{aligned}$$

It is clear that $I_5 \to 0$ as (x,λ) tends to (x_0,λ_0) in view of condition (c).

Applying Hölder's inequality to the integral I_4 (see, e.g., [25]), we have

$$\begin{aligned}
I_4 &= \left(\int\limits_{\Omega} |f(t)-f(x_0)|\, L_\lambda(t,x)dt\right)^p \\
&\leq \int\limits_{\Omega} |f(t)-f(x_0)|^p\, L_\lambda(t,x)dt \left(\int\limits_{\mathbb{R}} L_\lambda(t,x)dt\right)^{\frac{p}{q}} \\
&= I_{41} \times I_{42}.
\end{aligned}$$

Using condition (g), we see that $I_{42} \leq M^{\frac{p}{q}} < \infty$.

Now, let us find an appropriate inequality for the integral I_{41}. Splitting I_{41} into two parts, we have that

$$\begin{aligned}
&I_{41} \\
=\ & \int\limits_{\Omega} |f(t)-f(x_0)|^p\, L_\lambda(t,x)dt \\
=\ & \int\limits_{\langle x_0-\delta,x_0+\delta\rangle} |f(t)-f(x_0)|^p\, L_\lambda(t,x)dt + \int\limits_{\Omega\setminus\langle x_0-\delta,x_0+\delta\rangle} |f(t)-f(x_0)|^p\, L_\lambda(t,x)dt \\
=\ & I_{411}+I_{412}.
\end{aligned}$$

For the integral I_{412}, we may write

$$\begin{aligned}
I_{412} &= \int\limits_{\Omega\backslash\langle x_0-\delta,x_0+\delta\rangle} |f(t)-f(x_0)|^p L_\lambda(t,x)dt \\
&\le 2^p \left(\sup_{t\in\Omega\backslash\langle x_0-\delta,x_0+\delta\rangle} L_\lambda(t,x) \int\limits_{\Omega\backslash\langle x_0-\delta,x_0+\delta\rangle} |f(t)-f(x_0)|^p dt \right) \\
&\le 2^p \left(\sup_{t\in\Omega\backslash\langle x_0-\delta,x_0+\delta\rangle} L_\lambda(t,x) \left[\|f\|^p_{L_p(\Omega)} + |f(x_0)|^p |B-A| \right] \right).
\end{aligned}$$

Hence, by condition (d), $I_{412} \to 0$ as (x,λ) tends to (x_0,λ_0).

For the integral I_{411}, we may write

$$\begin{aligned}
I_{411} &= \int\limits_{\langle x_0-\delta,x_0+\delta\rangle} |f(t)-f(x_0)|^p L_\lambda(t,x)dt \\
&= \left\{ \int\limits_{x_0-\delta}^{x_0} + \int\limits_{x_0}^{x_0+\delta} \right\} |f(t)-f(x_0)|^p L_\lambda(t,x)dt \\
&= I_{4111} + I_{4112}.
\end{aligned}$$

Now, by definition of $\mu-$generalized Lebesgue point, for every $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$\int\limits_{x_0-h}^{x_0} |f(t)-f(x_0)|^p dt < \varepsilon^p \mu(h)$$

holds for every $0 < h \le \delta < \delta_0$. We define the new function as

$$G(t) = \int\limits_{t}^{x_0} |f(u)-f(x_0)|^p du.$$

Then, for every t satisfying $0 < x_0 - t \le \delta$, we have

$$|G(t)| \le \varepsilon^p \mu(x_0 - t).$$

We can write the integral I_{4111} as

$$\begin{aligned} |I_{4111}| &= \left| \int\limits_{x_0-\delta}^{x_0} |f(t)-f(x_0)|^p L_\lambda(t,x)dt \right| \\ &= \left| (LS) \int\limits_{x_0-\delta}^{x_0} L_\lambda(t,x) d\left[-G(t)\right] \right|, \end{aligned}$$

where (LS) denotes Lebesgue-Stieltjes integral.

Applying integration by parts method to the Lebesgue-Stieltjes integral, we have

$$\begin{aligned} |I_{4111}| &\leq \int\limits_{x_0-\delta}^{x_0} |G(t)| \, |d_t L_\lambda(t,x)| + |G(x_0-\delta)| \, L_\lambda(x_0-\delta,x) \\ &\leq \varepsilon^p \int\limits_{x_0-\delta}^{x_0} \mu(x_0-t) \, |d_t L_\lambda(t,x)| + \varepsilon^p \mu(\delta) L_\lambda(x_0-\delta,x) \\ &= \varepsilon^p \int\limits_{x_0-\delta}^{x_0} L_\lambda(t,x) \left| \{\mu(x_0-t)\}_t' \right| dt. \end{aligned}$$

Using similar technic, one may obtain the following inequality for the integral I_{4112}:

$$|I_{4112}| \leq \varepsilon^p \int\limits_{x_0}^{x_0+\delta} L_\lambda(t,x) \left| \{\mu(t-x_0)\}_t' \right| dt.$$

Combining above inequalities, we get

$$|I_{411}| \leq \varepsilon^p \int\limits_{x_0-\delta}^{x_0+\delta} L_\lambda(t,x) \left| \{\mu(|x_0-t|)\}_t' \right| dt.$$

Since the expression on the right hand side of the inequality is bounded by the hypothesis, we get the desired result for the case $1 < p < \infty$, i.e., the proof is completed. □

In the following theorem, $\Psi = \Omega = \mathbb{R}$.

THEOREM 4.2. *Let $x_0 \in \mathbb{R}$ be a $\mu-$generalized Lebesgue point of function $f \in L_p(\mathbb{R})$. Then,*

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |T_\lambda(f;x) - f(x_0)| = 0$$

on any set Z on which the function

$$\int\limits_{x_0-\delta}^{x_0+\delta} L_\lambda(t,x) \left|\{\mu(|x_0-t|)\}'_t\right| dt$$

is bounded as (x,λ) tends to (x_0,λ_0).

Proof. The proof of this theorem is quite similar to preceding one and we skip it. □

5. Weighted Pointwise Approximation

Throughout this section we assume that $f \notin L_p(\Omega)$, where $1 \le p < \infty$. Also, in the following theorem, $\Psi = \langle a,b\rangle$ and $\Omega = \langle A,B\rangle$ denote arbitrary closed, semi-closed or open bounded intervals in $\mathbb{R}$.

THEOREM 5.1. *Let w and L_λ be differentiable functions almost everywhere on $\mathbb{R}$ with respect to variable t such that the following inequality*

$$\frac{d}{dt}w(t)\frac{d}{dt}L_\lambda(t,x) > 0, \text{ for any fixed } x \in \mathbb{R}$$

holds. If $x_0 \in \Omega$ is a common $\mu-$generalized Lebesgue point of functions $f \in L_{p,w}(\Omega)$ and $w \in L_p(\Omega)$ then

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |T_\lambda(f;x) - f(x_0)| = 0$$

on any set Z on which the function

$$\int\limits_{x_0-\delta}^{x_0+\delta} w(t) L_\lambda(t,x) \left|\{\mu(|x_0-t|)\}'_t\right| dt$$

is bounded as (x,λ) tends to (x_0,λ_0).

Proof. We prove the theorem for the case $1 < p < \infty$. The proof of the case $p = 1$ is quite similar to the proof of the case $1 < p < \infty$ and it is omitted.

Let $\langle x_0-\delta, x_0+\delta\rangle \subset \Omega$, for fixed $\delta>0$, where x_0 is a common $\mu-$generalized Lebesgue point of functions $f \in L_{p,w}(\Omega)$ and $w \in L_p(\Omega)$. Using Theorem 4.1, we have

$$\left|\int\limits_{\Omega} K_\lambda(t,x,f(t))dt - f(x_0)\right|$$

$$\leq \int\limits_{\Omega}\left|\frac{f(t)}{w(t)}-\frac{f(x_0)}{w(x_0)}\right| w(t)L_\lambda(t,x)dt + \left|\int\limits_{\Omega} K_\lambda(t,x,\frac{f(x_0)}{w(x_0)}w(t))dt - f(x_0)\right|.$$

It is easy to see that the following inequality

$$\begin{aligned}\left|\int\limits_{\Omega} K_\lambda(t,x,f(t))dt - f(x_0)\right|^p &\leq 2^p\left(\int\limits_{\Omega}\left|\frac{f(t)}{w(t)}-\frac{f(x_0)}{w(x_0)}\right| w(t)L_\lambda(t,x)dt\right)^p \\ &+2^p\left|\int\limits_{\Omega} K_\lambda(t,x,\frac{f(x_0)}{w(x_0)}w(t))dt - f(x_0)\right|^p \\ &= 2^p(I_1+I_2)\end{aligned}$$

holds. By Theorem 4.1, $I_2 \to 0$ as (x,λ) tends to (x_0,λ_0). Applying Hölder's inequality to the integral I_1, we have

$$\begin{aligned} I_1 &\leq \int\limits_{\Omega}\left|\frac{f(t)}{w(t)}-\frac{f(x_0)}{w(x_0)}\right|^p w(t)L_\lambda(t,x)dt\left(\int\limits_{\Omega} w(t)L_\lambda(t,x)dt\right)^{\frac{p}{q}} \\ &= I_{11}\times I_{12}.\end{aligned}$$

In view of condition (g), $I_{12} \to (w(x_0)M)^{\frac{p}{q}} < \infty$ as (x,λ) tends to (x_0,λ_0). Let us find an appropriate inequality for the integral I_{11}. Splitting the integral I_{11} into two parts, we have

$$\begin{aligned} I_{11} &= \int\limits_{\Omega} \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right|^p w(t)L_\lambda(t,x)dt \\ &= \int\limits_{\langle x_0-\delta,x_0+\delta\rangle} \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right|^p w(t)L_\lambda(t,x)dt \\ &\quad + \int\limits_{\Omega\setminus\langle x_0-\delta,x_0+\delta\rangle} \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right|^p w(t)L_\lambda(t,x)dt \\ &= I_{111} + I_{112}. \end{aligned}$$

For the integral I_{112}, we may write

$$\begin{aligned} & I_{112} \\ &= \int\limits_{\Omega\setminus\langle x_0-\delta,x_0+\delta\rangle} \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right|^p w(t)L_\lambda(t,x)dt \\ &\leq 2^p \sup_{t\in\Omega\setminus\langle x_0-\delta,x_0+\delta\rangle} L_\lambda(t,x) \sup_{t\in\Omega\setminus\langle x_0-\delta,x_0+\delta\rangle} w(t) \int\limits_{\Omega\setminus\langle x_0-\delta,x_0+\delta\rangle} \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right|^p dt \end{aligned}$$

$$\leq 2^p \left(\sup_{t\in\Omega\setminus\langle x_0-\delta,x_0+\delta\rangle} L_\lambda(t,x) \sup_{t\in\Omega\setminus\langle x_0-\delta,x_0+\delta\rangle)} w(t) \left[\|f\|^p_{L_{p,w}(\Omega)} + \left|\frac{f(x_0)}{w(x_0)}\right|^p |B-A| \right] \right).$$

By the hypothesis, monotonicity behavior of w is similar to L_λ. Therefore, $w(t)$ is bounded on $\mathbb{R}\setminus\Omega$. Hence by condition (d), $I_{112} \to 0$ as (x,λ) tends to (x_0,λ_0).

Write

$$\begin{aligned} I_{111} &= \int\limits_{\langle x_0-\delta,x_0+\delta\rangle} \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right|^p w(t)K_\lambda(t,x)dt \\ &= \left\{ \int\limits_{x_0-\delta}^{x_0} + \int\limits_{x_0}^{x_0+\delta} \right\} \left|\frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)}\right|^p w(t)K_\lambda(t,x)dt \\ &= I_{1111} + I_{1112}. \end{aligned}$$

Now, by definition of $\mu-$generalized Lebesgue point, for every $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$\int_{x_0-h}^{x_0} \left| \frac{f(t)}{w(t)} - \frac{f(x_0)}{w(x_0)} \right|^p dt < \varepsilon^p \mu(h)$$

holds for every $0 < h \leq \delta < \delta_0$. We define the new function as

$$\widetilde{G}(t) = \int_{t}^{x_0} \left| \frac{f(u)}{w(u)} - \frac{f(x_0)}{w(x_0)} \right|^p du.$$

Then, for every t satisfying $0 < x_0 - t \leq \delta$ we have

$$\left| \widetilde{G}(t) \right| \leq \varepsilon^p \mu(x_0 - t).$$

Hence

$$\begin{aligned} |I_{1111}| &= \left| \int_{x_0-\delta}^{x_0} \left| \frac{f(u)}{w(u)} - \frac{f(x_0)}{w(x_0)} \right|^p w(t) L_\lambda(t,x) dt \right| \\ &= \left| (LS) \int_{x_0-\delta}^{x_0} w(t) L_\lambda(t,x) d\left[-\widetilde{G}(t) \right] \right|, \end{aligned}$$

where (LS) denotes Lebesgue-Stieltjes integral.

Applying integration by parts method to the Lebesgue-Stieltjes integral, we have

$$\begin{aligned} I_{1111} &= \int_{x_0-\delta}^{x_0} \left| \widetilde{G}(t) \right| d_t \left[w(t) L_\lambda(t,x) \right] + \widetilde{G}(x_0-\delta) w(x_0-\delta) L_\lambda(x_0-\delta, x) \\ &\leq \varepsilon^p \int_{x_0-\delta}^{x_0} \mu(x_0-t) \left| d_t \left[w(t) L_\lambda(t,x) \right] \right| + \varepsilon^p \mu(\delta) w(x_0-\delta) L_\lambda(x_0-\delta, x) \\ &= \varepsilon^p \int_{x_0-\delta}^{x_0} w(t) L_\lambda(t,x) \left| \{\mu(x_0-t)\}_t' \right| dt. \end{aligned}$$

Using similar technic, one may obtain the following inequality for the integral I_{1112}:

$$|I_{1112}| \le \varepsilon^p \int\limits_{x_0}^{x_0+\delta} w(t) L_\lambda(t,x) \left|\{\mu(t-x_0)\}'_t\right| dt.$$

Combining above inequalities, we get

$$|I_{111}| \le \varepsilon^p \int\limits_{x_0-\delta}^{x_0+\delta} w(t) L_\lambda(t,x) \left|\{\mu(|x_0-t|)\}'_t\right| dt.$$

Since the expression on the right hand side of the inequality is bounded by the hypothesis, we get the desired result for the case $1 < p < \infty$, i.e., the proof is completed. □

In the following theorem, $\Psi = \Omega = \mathbb{R}$.

Theorem 5.2. *Let w and L_λ be differentiable functions almost everywhere on $\mathbb{R}$ with respect to variable t such that the following inequality*

$$\frac{d}{dt} w(t) \frac{d}{dt} L_\lambda(t,x) > 0, \text{ for any fixed } x \in \mathbb{R}$$

holds. If $x_0 \in \mathbb{R}$ is a common $\mu-$generalized Lebesgue point of functions $f \in L_{p,w}(\mathbb{R})$ and $w \in L_p(\mathbb{R})$ then

$$\lim_{(x,\lambda)\to(x_0,\lambda_0)} |T_\lambda(f;x) - f(x_0)| = 0$$

on any set Z on which the function

$$\int\limits_{x_0-\delta}^{x_0+\delta} w(t) L_\lambda(t,x) \left|\{\mu(|x_0-t|)\}'_t\right| dt$$

is bounded as (x,λ) tends to (x_0,λ_0).

Proof. Following the same strategy as in Theorem 5.1, we have

$$\left| \int\limits_{\mathbb{R}} K_\lambda(t,x,f(t))dt - f(x_0) \right|^p \le$$

$$2^{2p}\left(w(x)M\right)^{\frac{p}{q}} \sup_{t\in\mathbb{R}\setminus\langle x_0-\delta,x_0+\delta\rangle} L_\lambda(t,x) \sup_{t\in\mathbb{R}\setminus\langle x_0-\delta,x_0+\delta\rangle)} w(t)\left\|f\right\|^p_{L_{p,w}(\mathbb{R})}$$

$$+2^{2p}\left(w(x)M\right)^{\frac{p}{q}} \sup_{t\in\mathbb{R}\setminus\langle x_0-\delta,x_0+\delta\rangle)} w(t)\left|\frac{f(x_0)}{w(x_0)}\right|^p \int\limits_{\mathbb{R}\setminus\langle x_0-\delta,x_0+\delta\rangle} L_\lambda(t,x)dt$$

$$+2^p\varepsilon^p\left(w(x)M\right)^{\frac{p}{q}} \int\limits_{x_0-\delta}^{x_0+\delta} w(t)L_\lambda(t,x)\left|\{\mu(|x_0-t|)\}'_t\right|dt$$

$$+2^p\left|\int\limits_{\mathbb{R}} K_\lambda(t,x,\frac{f(x_0)}{w(x_0)}w(t))dt-f(x_0)\right|^p.$$

The rest of the proof is clear by the hypotheses. □

6. Rate of Weighted Pointwise Convergence

In this section, two theorems concerning rate of pointwise convergence will be given.

THEOREM 6.1. *Suppose that the hypothesis of Theorem 5.1 is satisfied. Let*

$$\Delta(x,\lambda,\delta)=\int\limits_{x_0-\delta}^{x_0+\delta} w(t)L_\lambda(t,x)\left|\{\mu(|x_0-t|)\}'_t\right|dt,$$

where $0<\delta<\delta_0$, for a fixed (and finite!) positive number δ_0, and the following conditions are satisfied:

(i) $\Delta(x,\lambda,\delta)\to 0$ *as* $(x,\lambda)\to(x_0,\lambda_0)$, *for some* $\delta>0$.

(ii) For every $\xi>0$,

$$\sup_{t\in\mathbb{R}\setminus\langle x_0-\xi,x_0+\xi\rangle} L_\lambda(t,x)=o(\Delta(x,\lambda,\delta))$$

as $(x,\lambda)\to(x_0,\lambda_0)$

(iii) $\left|\int\limits_{\Omega} K_\lambda(t,x,\frac{f(x_0)}{\varphi(x_0)}\varphi(t))dt-f(x_0)\right|=o(\Delta(x,\lambda,\delta)).$

Then, at each common Lebesgue point of functions $f\in L_{p,\varphi}(\Omega)$ *and* $\varphi\in L_p(\Omega)$ *we have*

$$|T_\lambda (f; x) - f (x_0)|^p = o(\Delta(x, \lambda, \delta))$$

as $(x, \lambda) \to (x_0, \lambda_0)$.

Proof. By the hypotheses of Theorem 5.1, we have

$$\left| \int_\Omega K_\lambda(t, x, f(t))dt - f(x_0) \right|^p \leq$$

$$2^{2p} (w(x)M)^{\frac{p}{q}} \sup_{t\in\mathbb{R}\setminus\langle x_0-\delta,x_0+\delta\rangle} L_\lambda(t, x) \sup_{t\in\mathbb{R}\setminus\langle x_0-\delta,x_0+\delta\rangle)} w(t) \|f\|^p_{L_{p,w}(\Omega)}$$
$$+2^{2p} (w(x)M)^{\frac{p}{q}} \sup_{t\in\mathbb{R}\setminus\langle x_0-\delta,x_0+\delta\rangle)} w(t) \sup_{t\in\mathbb{R}\setminus\langle x_0-\delta,x_0+\delta\rangle} L_\lambda(t, x) \left|\frac{f(x_0)}{w(x_0)}\right|^p |B - A|$$
$$+2^p\varepsilon^p (w(x)M)^{\frac{p}{q}} \int_{x_0-\delta}^{x_0+\delta} w(t)L_\lambda(t, x) \left|\{\mu(|x_0 - t|)\}'_t\right| dt$$
$$+2^p \left| \int_\Omega K_\lambda(t, x, \frac{f(x_0)}{w(x_0)} w(t))dt - f(x_0)\right|^p .$$

The remaining part is obvious by the conditions $(i) - (iii)$. The proof of the case $p = 1$ is quite similar. Thus the proof is completed. □

THEOREM 6.2. *Suppose that the hypothesis of Theorem 5.2 is satisfied. Let*

$$\Delta(x, \lambda, \delta) = \int_{x_0-\delta}^{x_0+\delta} w(t)L_\lambda(t, x) \left|\{\mu(|x_0 - t|)\}'_t\right| dt,$$

where $0 < \delta < \delta_0$, *for a fixed (and finite!) positive number* δ_0, *and the following conditions are satisfied:*

(i) $\Delta(x, \lambda, \delta) \to 0$ *as* $(x, \lambda) \to (x_0, \lambda_0)$, *for some* $\delta > 0$.

(ii) For every $\xi > 0$,

$$\sup_{t\in\mathbb{R}\setminus\langle x_0-\xi,x_0+\xi\rangle} L_\lambda(t, x) = o(\Delta(x, \lambda, \delta))$$

as $(x, \lambda) \to (x_0, \lambda_0)$.

(iii) For every $\xi > 0$,

$$\int_{\mathbb{R}\setminus\langle x_0-\xi,x_0+\xi\rangle} L_\lambda(t,x)dt = o(\Delta(x,\lambda,\delta))$$

as $(x,\lambda) \to (x_0,\lambda_0)$.

(iv) $\left|\int\limits_{\mathbb{R}} K_\lambda(t,x,\frac{f(x_0)}{\varphi(x_0)}\varphi(t))dt - f(x_0)\right| = o(\Delta(x,\lambda,\delta))$.

Then, at each common Lebesgue point of functions $f \in L_{p,\varphi}(\mathbb{R})$ *and* $\varphi \in L_p(\mathbb{R})$ *we have*

$$|T_\lambda(f;x) - f(x_0)|^p = o(\Delta(x,\lambda,\delta))$$

as $(x,\lambda) \to (x_0,\lambda_0)$.

Proof. Imposing the conditions (i)-(iv) on the resulting inequality of the proof of Theorem 5.2, we obtain the desired result. The proof of the case $p = 1$ is quite similar. Thus the proof is completed. □

7. Concluding Remark

In this paper, weighted approximation properties of nonlinear non-convolution type singular integral operators are studied. For this aim, the class of kernel functions, called *Class* A_w, is defined. Therefore, main theorems are presented as Theorem 5.1 and Theorem 5.2.

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