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ON A CLASSIFICATION OF WARPED PRODUCT SPACES WITH GRADIENT RICCI SOLITONS

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Abstract. In this paper, we study Ricci solitons, gradient Ricci solitons in the warped product spaces and gradient Yamabe solitons in the Riemannian product spaces. We obtain the necessary and sufficient conditions for the Riemannian product spaces to be Ricci solitons. Moreover we classify the warped product space which admit gradient Ricci solitons under some conditions of the potential function.

1. Introduction

A Riemannian metric g on a complete Riemannian manifold M is called a Ricci soliton if there exists a smooth vector field X such that the Ricci curvature tensor satisfies

$$Ric + \frac{1}{2}\mathfrak{L}_X g = \rho g \tag{1}$$

for some constant ρ, where $\mathfrak{L}_X$ is the Lie derivative with respect to X [2, 3, 5, 6, 9]. It is said that (M, g) or M is a Ricci soliton if the metric g on M is a Ricci soliton. Evidently we see that an Einstein metric becomes a Ricci soliton, but the converse is not true. The Ricci

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soliton is called shrinking if $\rho > 0$, steady if $\rho = 0$ and expanding if $\rho < 0$. The metric of a Ricci soliton is useful in not only physics but also mathematics, and often referred as quasi-Einstein [**?**]. If $X = \nabla h$ for some function h on M, then M is called a gradient Ricci soliton. The function h above is called the potential function. In this case, the equation (1) can be rewritten as

$$Ric + Hess\ h = \rho g. \tag{2}$$

It is well known that when $\rho \leq 0$ all compact solitons are necessarily Einstein [6], and a Ricci soliton on a compact manifold has a constant curvature in 2-dimension [**?**] as well as in 3-dimension [8]. Moreover a Ricci soliton on a compact manifold is a gradient Ricci soliton [9], and a compact shrinking soliton is always gradient [13].

In [11], we studied the gradient Ricci soliton in the warped product space and proved that the base space of the warped product space with a gradient Ricci soliton to be a gradient Ricci soliton or an Einstein space is determined by the second derivative of the warping function. As far as we know, there are no results for the warped products of two Riemannian manifolds with general dimension and gradient Ricci soliton related to potential function. In this sense, the considerations of the warped products with gradient Ricci soliton satisfying some conditions related to potential function are meaningful. Related these problems, we get some useful results and classification theorem (see Theorem 3.3).

A Riemannian metric g on a Riemannian manifold M is called a Yamabe soliton if there exist a smooth vector field X and a constant ρ such that

$$(r - \rho)g = \frac{1}{2}\mathfrak{L}_X g, \tag{3}$$

where r is the scalar curvature of M. In particular, if $X = \nabla h$ for some smooth function h, we call it the gradient Yamabe soliton [**?**]. The function h above is called the potential function. In this case, the equation (3) can be rewritten as

$$(r - \rho)g = \nabla^2 h. \tag{4}$$

In Chapter 4, we studied gradient Yamabe soliton in $M = R^1 \times B$ for a Riemannian manifold B, and we get the necessary and sufficient condition for M admits a gradient Yamabe soliton.

2. Ricci solitons in the Riemannian product manifolds

Let (B, g) be an m-dimensional Riemannian manifold with a metric g and let $M = R\times B$ be the product Riemannian manifold with the metric $\tilde{g} = \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}$. Then the Ricci curvature tensors $\tilde{S}$ and S of M and B, respectively are given by $S_{ab} = S_{ab}$ and the others are zero, where the range of indices $a, b, c, \cdots$ is $\{2, 3, \cdots, m+1\}$.

Suppose that B is a Ricci soliton. Then there exists a smooth vector field $U = (\xi_2, \cdots, \xi_{m+1})$ on B such that

$$S_{ab} = \rho g_{ab} - \frac{1}{2}(\nabla_a \xi_b + \nabla_b \xi_a),$$

for some constant ρ.

If we take $\widetilde{\rho} = \rho$ and $\widetilde{U} = (\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_{m+1}) = (\rho t, \xi_2, \cdots, \xi_{m+1})$, then we obtain

$$\begin{aligned} \tilde{S}_{ab} &= S_{ab} = \rho g_{ab} - \tfrac{1}{2}(\nabla_a \xi_b + \nabla_b \xi_a) = \tilde{\rho}\tilde{g}_{ab} - \tfrac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a), \\ \tilde{S}_{a1} &= 0 = -\tfrac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_1 + \tilde{\nabla}_1 \tilde{\xi}_a), \\ \tilde{S}_{11} &= 0 = \tilde{\rho} - \tfrac{1}{2}(\tilde{\nabla}_1 \tilde{\xi}_1 + \tilde{\nabla}_1 \tilde{\xi}_1). \end{aligned} \tag{5}$$

That is, M becomes a Ricci soliton.

Conversely, if we suppose that M is a Ricci soliton, then there exists a smooth vector field $\tilde{V} = (\widetilde{\xi}_1, \widetilde{\xi}_2, \cdots, \widetilde{\xi}_{m+1})$ on M such that

$$\tilde{S}_{ij} = \tilde{\rho}\tilde{g}_{ij} - \frac{1}{2}(\tilde{\nabla}_i \tilde{\xi}_j + \tilde{\nabla}_j \tilde{\xi}_i)$$

for some constant function $\tilde{\rho}$ on M, where the range of indices i,j,k,$\cdots$ is $\{1, 2, 3, \cdots, m+1\}$. So we have

$$\begin{aligned} \tilde{S}_{ab} &= \tilde{\rho}\tilde{g}_{ab} - \tfrac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a) = S_{ab}, \\ \tilde{S}_{a1} &= -\tfrac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_1 + \tilde{\nabla}_1 \tilde{\xi}_a) = -\tfrac{1}{2}(\partial_a \tilde{\xi}_1 + \partial_1 \tilde{\xi}_a) = 0, \\ \tilde{S}_{11} &= \tilde{\rho} - \tfrac{1}{2}(\tilde{\nabla}_1 \tilde{\xi}_1 + \tilde{\nabla}_1 \tilde{\xi}_1) = \tilde{\rho} - \partial_1 \tilde{\xi}_1 = 0. \end{aligned} \tag{6}$$

From the third equation of (6), we can put

$$\tilde{\xi}_1 = \tilde{\rho} t + h(x_2, \cdots, x_{m+1}) \tag{7}$$

for $t \in R$ and some function h on B. Using the second equation of (6) and (7), we see that $\tilde{\xi}_a$ is of the form

$$\tilde{\xi}_a \;=\; -h_a t + k^a, \tag{8}$$

where $h_a = \partial_a h$ and $k^a = k^a(x_2, \cdots, x_{m+1})$ is a function on B corresponding to $\tilde{\xi_a}$. Hence, the vector field $\tilde{V}$ on M is given by

$$\tilde{V} \;=\; (\tilde{\rho}t + h, -h_2 t + k^2, \cdots, -h_{m+1}t + k^{m+1}).$$

From the first equation of (6) and the fact that $S_{ab} - \rho g_{ab}$ depends only on B, we find that the equation $S_{ab} - \rho g_{ab} = -\frac{1}{2}(\nabla_a \xi_b + \nabla_b \xi_a)$ holds for $\rho = \tilde{\rho}$ and $\xi_a = k^a$. If we take the vector field $V \;=\; (\xi_2, \cdots, \xi_{m+1})$ on B, then we see that B is a Ricci soliton. Thus, we have

THEOREM 2.1. *$M = R \times B$ is a Ricci soliton if and only if B is a Ricci soliton.*

Next, consider the case of $M = R^2 \times B$ be the product Riemannian manifold with the Riemannian metric $\tilde{g} = \begin{pmatrix} \delta_{uv} & 0 \\ 0 & g_{ab} \end{pmatrix}$, where the range of indices u, v is $\{1, 2\}$ and the range of indices a,b,c,$\cdots$ is $\{3, 4, \cdots, m+2\}$. Then, the Ricci curvature tensors $\tilde{S}$ and S of M and B, respectively, are given by $\tilde{S}_{ab} = S_{ab}$ and the others are zero.

Suppose that B is a Ricci soliton. Then there exists a smooth vector field $U = (\xi_3, \cdots, \xi_{m+2})$ on B such that $S_{ab} - \rho g_{ab} = -\frac{1}{2}(\nabla_a \xi_b + \nabla_b \xi_a)$ for some constant ρ. Take $\widetilde{\rho} = \rho$ and $\tilde{U} = (\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_{m+2}) = (\rho t_1, \rho t_2, \xi_3, \cdots, \xi_{m+2})$, then we obtain

$$\begin{aligned} \tilde{S}_{ab} &= S_{ab} = \rho g_{ab} - \tfrac{1}{2}(\nabla_a \xi_b + \nabla_b \xi_a) = \tilde{\rho}\tilde{g}_{ab} - \tfrac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_b + \tilde{\nabla}_b \tilde{\xi}_a), \\ \tilde{S}_{au} &= 0 = -\tfrac{1}{2}(\tilde{\nabla}_a \tilde{\xi}_u + \tilde{\nabla}_u \tilde{\xi}_a), \\ \tilde{S}_{uv} &= 0 = \tilde{\rho}\delta_{uv} - \tfrac{1}{2}(\tilde{\nabla}_u \tilde{\xi}_v + \tilde{\nabla}_v \tilde{\xi}_u). \end{aligned} \tag{9}$$

Hence we can state, if B is a Ricci soliton, then $M = R^2 \times B$ is a Ricci soliton.

Conversely, suppose that $M = R^2 \times B$ is a Ricci soliton. Then, we get

$$\begin{aligned} S_{ab} = \tilde{S}_{ab} = \tilde{\rho} g_{ab} - \tfrac{1}{2}\left(\partial_a \tilde{\xi}_b + \partial_b \tilde{\xi}_a - 2\left\{ \begin{matrix} c \\ ab \end{matrix} \right\} \tilde{\xi}_c\right), \\ \partial_a \tilde{\xi}_u + \partial_u \tilde{\xi}_a = 0, \\ \tilde{\rho}\delta_{uv} - \tfrac{1}{2}(\partial_u \tilde{\xi}_v + \partial_v \tilde{\xi}_u) = 0. \end{aligned} \tag{10}$$

for some constant $\tilde{\rho}$ on M, where $\tilde{\xi}_i$ is a component of the smooth vector field $\tilde{V}$ on M and the range of indices $i, j, k, \cdots$ is $\{1, 2, 3, \cdots, m+2\}$. Considering the case $u = v$ in the third equation of (10), we obtain

$$\tilde{\xi}_u = \tilde{\rho} x_u + h^u_{(u)}, \tag{11}$$

where $h^u_{(u)}$ is a function on M which corresponds to $\tilde{\xi}_u$ and the symbol (u) means having no x_u-variable.

From the third equation of (10) and (11), we obtain

$$\partial_u h^v_{(v)} = -\partial_v h^u_{(u)}, \tag{12}$$

from which we see that $\partial_u h^v_{(v)}$ is a function on B. Thus, we can denote

$$\partial_u h^v_{(v)} = H_{(u,v)}, \tag{13}$$

where $H_{(u,v)}$ means a function H having no x_u-variable and x_v-variable.

By use of the equations (11) and (13), we obtain

$$h^u_{(u)} = -H_{(u,v)} x_v + K^u_{(1,2)}, \tag{14}$$

and

$$\tilde{\xi}_u = \tilde{\rho} x_u - H_{(u,v)} x_v + K^u_{(1,2)}, (u \neq v), \tag{15}$$

from the equations (11) and (14).

The second equation of (10) and (15) give rise to

$$\tilde{\xi}_a = -\partial_a K^1_{(1,2)} x_1 - \partial_a K^2_{(1,2)} x_2 + L^a_{(1,2)}, \tag{16}$$

and the first equation of (10) gives

$$S_{ab} - \tilde{\rho} g_{ab} = -\frac{1}{2}(\partial_a \tilde{\xi}_b + \partial_b \tilde{\xi}_a - 2 \left\{ \begin{array}{c} c \\ ab \end{array} \right\} \tilde{\xi}_c).$$

Moreover, we get

$$\tilde{V} = (\tilde{\rho} x_1 - H_{(1,2)} x_2 + K^1_{(1,2)},\ \tilde{\rho} x_2 - H_{(2,1)} x_1 + K^2_{(1,2)},$$
$$-\partial_3 K^1_{(1,2)} x_1 - \partial_3 K^2_{(1,2)} x_2 + L^3_{(1,2)}, \cdots, -\partial_{m+2} K^1_{(1,2)} x_1 - \partial_{m+2} K^2_{(1,2)} x_2 + L^{m+2}_{(1,2)})$$

by means of the equations (15) and (16).

If we take $V = (\xi_3,\ \cdots,\ \xi_{m+2})$ such that $\xi_a = L^a_{(1,2)}$, then we get

$$S_{ab} - \rho g_{ab} = -\frac{1}{2}(\nabla_a L^b_{(1,2)} + \nabla_b L^a_{(1,2)}),$$

for $\rho = \tilde{\rho}$. Hence we see that if $M = R^2 \times B$ is a Ricci soliton, then B is a Ricci soliton. Thus we get

Theorem 2.2. *$M = R^2 \times B$ is a Ricci soliton if and only if B is a Ricci soliton.*

In general, if we assume that the Riemannian products $M = B \times F$ is a Ricci soliton, then there exists vector field $X = (\tilde{\xi}_1, \cdots, \tilde{\xi}_n, \tilde{\xi}_{n+1}, \cdots, \tilde{\xi}_{n+p})$ on M and constant $\tilde{\rho}$ such that $\tilde{S}_{ij} = \tilde{\rho}\tilde{g}_{ij} - (\tilde{\nabla}_i\tilde{\xi}_j + \tilde{\nabla}_j\tilde{\xi}_i)/2$, that is

$$
\begin{aligned}
S_{ab} &= \tilde{\rho} g_{ab} - (\tilde{\nabla}_a\tilde{\xi}_b + \tilde{\nabla}_b\tilde{\xi}_a)/2,\\
&\tilde{\nabla}_a\tilde{\xi}_x + \tilde{\nabla}_x\tilde{\xi}_a = 0,\\
\bar{S}_{xy} &= \tilde{\rho} g_{xy} - (\tilde{\nabla}_x\tilde{\xi}_y + \tilde{\nabla}_y\tilde{\xi}_x)/2.
\end{aligned}
$$

If we take $\rho = \tilde{\rho}$ and $X_B = (\tilde{\xi}_1, \cdots, \tilde{\xi}_n)$, then we see that B is a Ricci soliton. Similarly we find F is Ricci soliton with $\bar{\rho} = \tilde{\rho}$ and $X_F = (\tilde{\xi}_{n+1}, \cdots, \tilde{\xi}_{n+p})$.

Conversely, if we assume that B is a Ricci soliton with $X_B = (\xi_1, \cdots, \xi_n)$ and constant ρ on B , and F is a Ricci soliton with $X_F = (\xi_{n+1}, \cdots, \xi_{n+p})$ and constant $\bar{\rho}$ on F. then we can express

$$
S_{ab} = \rho g_{ab} - (\nabla_a\xi_b + \nabla_b\xi_a)/2, \ \bar{S}_{xy} = \bar{\rho} g_{xy} - (\bar{\nabla}_x\xi_y + \bar{\nabla}_y\xi_x)/2.
$$

Now we take $\tilde{\rho} = \rho = \bar{\rho}$ and $X = (X_B, X_F)$ on M, then we have

$$
\begin{aligned}
\tilde{S}_{ab} &= S_{ab} = \tilde{\rho}\tilde{g}_{ab} - (\tilde{\nabla}_a\xi_b + \tilde{\nabla}_b\xi_a)/2,\\
\tilde{S}_{ax} &= \ 0 \ = \tilde{\rho}\tilde{g}_{ax} - (\tilde{\nabla}_a\xi_x + \tilde{\nabla}_x\xi_a)/2,\\
\tilde{S}_{xy} &= \bar{S}_{xy} = \tilde{\rho}\tilde{g}_{xy} - (\tilde{\nabla}_x\xi_y + \tilde{\nabla}_y\xi_x)/2,
\end{aligned}
$$

that is, $\tilde{S}_{ij} = \tilde{\rho}\tilde{g}_{ij} - (\tilde{\nabla}_i\xi_j + \tilde{\nabla}_j\xi_i)/2$. Hence we see that $M = B \times F$ is a Ricci soliton. Thus we have

Theorem 2.3. *Let $M = B \times F$ be a Riemannian product of B and F. Then M is a Ricci soliton if and only if B and F are Ricci solitons with $\rho = \bar{\rho}$.*

3. Gradient Ricci solitons in the warped product spaces

Let (B, g) and $(F, \bar{g})$ be n and p-dimensional Riemannian manifolds with Riemanian metric g and $\bar{g}$ respectively. Then the warped product space $M = B \times_f F$ with a warping function f has the Riemannian metric $G = \begin{pmatrix} g & 0 \\ 0 & f^2\bar{g} \end{pmatrix}$. Hence the components of Ricci tensors are given by [1, 10, 11]

$$\tilde{S}_{ab} = S_{ab} - \frac{p}{f}\nabla_a f_b,$$
$$\tilde{S}_{ax} = 0,$$
$$\tilde{S}_{xy} = \bar{S}_{xy} - f(\triangle f)\bar{g}_{xy} - (p-1)\|\|f_e\|\|^2\bar{g}_{xy},$$

where $\tilde{S}$, S and $\bar{S}$ are the Ricci tensors of M, B and F respectively, and $\triangle f$ is the Laplacian of f for g.

Suppose that $M = B \times_f F$ is a gradient Ricci solution. Then, we have

$$\tilde{S}_{ab} = \tilde{\rho}\tilde{g}_{ab} - \tilde{\nabla}_a\tilde{\nabla}_b h = \tilde{\rho}g_{ab} - \nabla_a\nabla_b h,$$
$$\tilde{S}_{ax} = \tilde{\rho}\tilde{g}_{ax} - \tilde{\nabla}_a\tilde{\nabla}_x h = -\partial_a h_x + \frac{f_a}{f}h_x,$$
$$\tilde{S}_{xy} = \tilde{\rho}\tilde{g}_{xy} - \tilde{\nabla}_x\tilde{\nabla}_y h = \tilde{\rho}f^2\bar{g}_{xy} - (\bar{\nabla}_x\bar{\nabla}_y h + ff^c h_c\bar{g}_{xy}),$$

for some constant $\tilde{\rho}$ and some function h on M.

Then we have

$$S_{ab} - \frac{p}{f}\nabla_a f_b = \tilde{\rho}g_{ab} - \nabla_a h_b, \tag{17}$$

$$\partial_a h_x = \frac{f_a}{f}h_x, \tag{18}$$

$$\bar{S}_{xy} = (\tilde{\rho}f^2 + f(\triangle f) + (p-1)\|\|f_e\|\|^2 - ff^c h_c)\bar{g}_{xy} - \bar{\nabla}_x\bar{\nabla}_y h. \tag{19}$$

Assume that the partial derivative of h with respect to all x vanishes, then h becomes a function on B. We see that F is an Einstein space because the coefficient of $\bar{g}_{xy}$ in (19) is constant along F. Thus we have

THEOREM 3.1. *Let $M = B \times_f F$ be a gradient Ricci soliton with a potential function h. If the partial derivative of h with respect to all x vanishes, then F becomes an Einstein space.*

If $h_x \neq 0$ for all x, then equation (18) becomes $\frac{\partial_a h_x}{h_x} = \frac{f_a}{f}$. Hence $\partial_a(lnh_x - lnf) = 0$, that is we can $ln\frac{h_x}{f} = l$ for some function l on F. Therefore $h_x = fe^l$, which induces $h = fk$ for some function k on F. Since $h_x \neq 0$ for all x and f depends on B, $k_x \neq 0$. Hence we have $h_a = f_a k$, $\nabla_b h_a = k\nabla_b f_a$ and the equation (17) becomes

$$S_{ab} - \frac{p}{f}\nabla_a f_b - \tilde{\rho} g_{ab} = k\nabla_b f_a. \tag{20}$$

If we assume that $\nabla_b f_a \neq 0$ for some a and b , then it leads to a contradiction in the equation (20) because the left hand side of (20) only depends on B and k the right hand side depends on B and F. Hence we see that $\nabla_b f_a = 0$ for all a and b and that B becomes an Einstein space from the equation (20). Moreover $\nabla_b f_a = 0$ implies Laplacian of f equal to zero. So if B is compact, then f becomes a constant, that is, M is the Riemannian product space. Thus we have

THEOREM 3.2. *Let $M = B \times_f F$ be a gradient Ricci soliton with a potential function h. If $h_x \neq 0$ for all x, then B becomes an Einstein space. In this case $\nabla_b f_a = 0$ for all a and b, and M is the Riemannian product space if B is compact.*

On the other hand, if the partial derivative $h_a = 0$ for all a, then h only depends on F and $f_a h_x = 0$ from (18). Hence f is a constant function or h is a constant function. The fact f is a constant means that M is the Riemannian product of B and F, moreover B is Einstein and F becomes a gradient Ricci soliton due to (17) and (19). If h is a constant function, then we see that the Ricci curvature on B has the form $S_{ab} = \tilde{\rho} g_{ab} + \frac{p}{f}\nabla_a f_b$ and F is Einstein from (19). Moreover if f and h are constants, then the Ricci curvatures of B and F are given by $S_{ab} = \tilde{\rho} g_{ab}$ and $\bar{S}_{xy} = \tilde{\rho} f^2 \bar{g}_{xy}$, respectively. Since the metric G on M is given by $G = \begin{pmatrix} g & 0 \\ 0 & f^2\bar{g} \end{pmatrix}$, we see that $\tilde{S} = \tilde{\rho} G$, that is, M is Einstein. Thus we have

THEOREM 3.3. *Let $M = B \times_f F$ be a gradient Ricci soliton with a potential function h. If the partial derivative $h_a = 0$ for all a, then one of the following three cases occurs.*
(a) M is the Riemannian product of an Einstein space and a gradient Ricci soliton space.
(b) F is Einstein.
(c) M is the Riemannian product of two Einstein spaces, and moreover M is an Einstein space.

4. Gradient Yamabe solitons in the Riemannian product manifolds

At first, we consider the gradient Yamabe soliton in the product space $M = R \times B$ with a Riemannian metric $\tilde{g}$ and a potential function h, where (B, g) is an m-dimensional Riemannian manifold. Then we see that

$$\begin{aligned} (\tilde{r} - \tilde{\rho}) &= \partial_1 h_1, \\ \partial_1 h_a &= 0, \\ (\tilde{r} - \tilde{\rho}) g_{ab} &= \nabla_a \nabla_b h, \\ \tilde{r} &= r, \end{aligned} \tag{21}$$

where $\tilde{r}$ and r are scalar curvatures of M and B respectively, and the range of indices $a, b, c, \cdots$ is $\{2, 3, \cdots, m+1\}$. From the second equation of (21), we see that the potential function h is decomposed into $h(t, x_1, x_2, \cdots, x_m) = l(t) + k(x_1, \cdots, x_m)$ for functions l on R and k on B. Then the third equation of (21) can be rewritten as $(\tilde{r} - \tilde{\rho}) g_{ab} = \nabla_a \nabla_b k$, that is, B is a gradient Yamabe soliton. Moreover, from (21), we see that $\partial_1 l_1 = \partial_1 h_1 = \tilde{r} - \tilde{\rho} = \frac{1}{m} g^{ab} \nabla_a \nabla_b h$. Since $\partial_1 l_1$ only depends on R and the right hand side is a quantity of B, $\tilde{r} - \tilde{\rho}$ becomes constant, that is, $r = \tilde{r}$ is constant.

Conversely, let (B, g) be a gradient Yamabe soliton with a constant scalar curvature r. Then $(r - \rho) g_{ab} = \nabla_a \nabla_b k$ with a potential function k and $\tilde{r} = r$ is constant. Take the function h on M as $h(t, x_1, x_2, \cdots, x_m) = (r - \rho)\frac{t^2}{2} + k(x_1, \cdots, x_m)$ and $\tilde{\rho} = \rho$. Then we get
$(\tilde{r} - \tilde{\rho}) = \partial_1 h_1$, $\partial_1 h_a = 0$,
$(\tilde{r} - \tilde{\rho}) \tilde{g}_{ab} = \tilde{\nabla}_a \tilde{\nabla}_b h$,
that is, $M = R^1 \times B$ becomes a gradient Yamabe soliton. Thus we have

THEOREM 4.1. *$M = R^1 \times B$ is a gradient Yamabe soliton if and only if B is a gradient Yamabe soliton with a constant scalar curvature.*

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