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STABILITY IN NONLINEAR NEUTRAL LEVIN-NOHEL INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper we use the Krasnoselskii-Burton's fixed point theorem to obtain asymptotic stability and stability results about the zero solution for the following nonlinear neutral Levin-Nohel integro-differential equation

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s)g\left(x(s)\right)ds + c(t)x'(t-\tau(t)) = 0.$$

The results obtained here extend the work of Mesmouli, Ardjouni and Djoudi [20].

1. Introduction

The Lyapunov direct method has been very effective in establishing stability results and the existence of periodic solutions for wide variety of ordinary, functional and partial differential equations. Nevertheless, in the application of Lyapunov's direct method to problems of stability in delay differential equations, serious difficulties occur if the delay is unbounded or if the equation has unbounded terms. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Zhang and others began a study in which

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they noticed that some of this difficulties vanish or might be overcome by means of fixed point theory (see [1]– [22], [24]). The fixed point theory does not only solve the problem on stability but has other significant advantage over Lyapunov's direct method. The conditions of the former are often average but those of the latter are usually pointwise (see [8]).

In paper, we consider the following nonlinear neutral Levin-Nohel integro-differential equation with variable delay

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s) g\left(x(s)\right) ds + c(t) x'(t-\tau(t)) = 0, \; t \geq t_0, \tag{1}$$

with an assumed initial condition

$$x\left(t\right) = \phi\left(t\right), \; t \in \left[m\left(t_0\right), t_0\right],$$

where $\phi \in C\left(\left[m\left(t_0\right), t_0\right], \mathbb{R}\right)$ and

$$m\left(t_0\right) = \inf\left\{t - \tau\left(t\right) : t \in \left[t_0, \infty\right)\right\}.$$

Throughout this paper, we assume that $c \in C^1\left(\left[t_0, \infty\right), \mathbb{R}\right)$, $a \in C([t_0, \infty) \times [m\left(t_0\right), \infty), \mathbb{R}_+)$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous with respect to its argument. We assume that $g\left(0\right) = 0$ and $\tau \in C^2\left(\left[t_0, \infty\right), \mathbb{R}^+\right)$ such that

$$\tau'\left(t\right) \neq 1, \; t \in \left[t_0, \infty\right). \tag{2}$$

Our purpose here is to use the Krasnoselskii-Burton's fixed point theorem to show the asymptotic stability and stability of the zero solution for (1). In the special case $c = 0$, Mesmouli, Ardjouni and Djoudi [20] show the zero solution of (1) is asymptotically stable with a necessary and sufficient condition by using the contraction mapping theorem in a weighted Banach space. The results presented in this paper extend the main results in [20].

2. The inversion and the fixed point theorem

One crucial step in the investigation of an equation using fixed point theory involves the construction of a suitable fixed point mapping. For that end we must invert (1) to obtain an equivalent integral equation from which we derive the needed mapping. During the process, an integration by parts has to be performed on the neutral term $x'(t - \tau(t))$. Unfortunately, when doing this, a derivative $\tau'(t)$ of the delay appears on the way, and so we have to support it.

LEMMA 2.1. *Suppose that (2) holds. Then x is a solution of equation (1) if and only if*

$$x(t) = (\phi(t_0) + \gamma(t_0)\phi(t_0 - \tau(t_0)))\, e^{-\int_{t_0}^{t} A(z)dz} + \int_{t_0}^{t} \left(\int_{s-\tau(s)}^{s} a(s,u)\,(Gx)\,(u)du \right) e^{-\int_{s}^{t} A(z)dz} ds - \gamma(t)x(t-\tau(t)) - \int_{t_0}^{t} [L_x(s) - \mu(s)x(s-\tau(s))]\, e^{-\int_{s}^{t} A(z)dz} ds,\ t \geq t_0, \tag{3}$$

where

$$L_x(t) = \int_{t-\tau(t)}^{t} a(t,s) \left(\int_{s}^{t} \left(\int_{u-\tau(u)}^{u} a(u,v)x(v)dv - r(u)x(u-\tau(u)) \right) du + \gamma(t)x(t-\tau(t)) - \gamma(s)x(s-\tau(s)) \right) ds \tag{4}$$

$$r(t) = \frac{c'(t)(1-\tau'(t)) + \tau''(t)c(t)}{(1-\tau'(t))^2},\ \gamma(t) = \frac{c(t)}{1-\tau'(t)}, \tag{5}$$

$$(Gx)(t) = x(t) - g(x(t)), \tag{6}$$

and

$$\mu(t) = \frac{(c'(t) + c(t)A(t))(1-\tau'(t)) + \tau''(t)c(t)}{(1-\tau'(t))^2},\ A(t) = \int_{t-\tau(t)}^{t} a(t,s)ds. \tag{7}$$

Proof. Let x be a solution of (1). Rewrite (1) as

$$x'(t) + \int_{t-\tau(t)}^{t} a(t,s)x(s)ds - \int_{t-\tau(t)}^{t} a(t,s)\,(x(s) - g(x(s)))\,ds + c(t)x'(t-\tau(t)) = 0,\ t \geq t_0.$$

Obviously, we have

$$x(s) = x(t) - \int_{s}^{t} x'(u)du.$$

Inserting this relation into (1), we get

$$
\begin{aligned}
&x'(t) + \int_{t-r(t)}^{t} a(t,s) \left(x(t) - \int_{s}^{t} x'(u)du \right) ds \\
&\quad - \int_{t-\tau(t)}^{t} a(t,s)(Gx)(s)\, ds + c(t)x'(t-\tau(t)) \;=\; 0,\;\; t \;\geq\; t_0,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&x'(t) + x(t) \int_{t-\tau(t)}^{t} a(t,s)ds - \int_{t-\tau(t)}^{t} a(t,s) \left(\int_{s}^{t} x'(u)du \right) ds \\
&- \int_{t-\tau(t)}^{t} a(t,s)(Gx)(s)\, ds + c(t)x'(t-\tau(t)) \;=\; 0,\;\; t \;\geq\; t_0.
\end{aligned}
$$

After substituting x' from (1), we obtain

(8)

$$
\begin{aligned}
&x'(t) + x(t) \int_{t-\tau(t)}^{t} a(t,s)ds \\
&\quad + \int_{t-\tau(t)}^{t} a(t,s) \left(\int_{s}^{t} \left(\int_{u-\tau(u)}^{u} a(u,v)x(v)dv + c(u)x'(u-\tau(u)) \right) du \right) ds \\
&\quad - \int_{t-\tau(t)}^{t} a(t,s)(Gx)(s)\, ds + c(t)x'(t-\tau(t)) = 0,\;\; t \geq t_0.
\end{aligned}
$$

By performing the integration by parts, we have

$$
\begin{aligned}
&\int_{s}^{t} c(u)x'(u-\tau(u))du \\
&= \int_{s}^{t} \frac{c(u)}{1-\tau'(u)} dx(u-\tau(u)) \\
&= \gamma(t)\, x(t-\tau(t)) - \gamma(s)\, x(s-\tau(s)) - \int_{s}^{t} r(u)x(u-\tau(u))du, \qquad (9)
\end{aligned}
$$

where r and γ are given by (5). After substituting (9) into (8), we have

$$
\begin{aligned}
&x'(t) + A(t)x(t) + L_x(t) \\
&- \int_{t-\tau(t)}^{t} a(t,s)(Gx)(s)\, ds + c(t)x'(t-\tau(t)) \;=\; 0,\;\; t \;\geq\; t_0,
\end{aligned}
$$

where A and L_x are given by (7) and (4), respectively. By the variation of constants formula, we get

$$x(t) = \phi(t_0)e^{-\int_{t_0}^{t} A(z)dz} + \int_{t_0}^{t} \left(\int_{s-\tau(s)}^{s} a(s,u)\,(Gx)\,(u)du \right) e^{-\int_s^t A(z)dz} ds$$

$$(10) \quad - \int_{t_0}^{t} \left[L_x(s) + c(s)x'(s-\tau(s)) \right] e^{-\int_s^t A(z)dz} ds, \ t \geq t_0.$$

Letting

$$\int_{t_0}^{t} c(s)x'(s-\tau(s))e^{-\int_s^t A(z)dz} ds = \int_{t_0}^{t} \frac{c(s)}{1-\tau'(s)} e^{-\int_s^t A(z)dz} dx(s-\tau(s)).$$

By using the integration by parts, we obtain

$$\int_{t_0}^{t} c(s)x'(s-\tau(s))e^{-\int_s^t A(z)dz} ds$$
$$= \frac{c(t)}{1-\tau'(t)} x(t-\tau(t)) - \frac{c(t_0)}{1-\tau'(t_0)} x(t_0 - \tau(t_0))e^{-\int_{t_0}^{t} A(z)dz}$$
$$(11) \quad - \int_{t_0}^{t} \mu(s)x(s-\tau(s))e^{-\int_s^t A(z)dz} ds,$$

where μ is given by (7). Finally, we obtain (3) by substituting (11) in (10). Since each step is reversible, the converse follows easily. This completes the proof. $\square$

Burton studied the theorem of Krasnoselskii and observed (see [9]) that Krasnoselskii result can be more interesting in applications with certain changes and formulated the Theorem 2.4 below (see [9] for its proof).

Definition 2.2. Let (M,d) be a metric space and $F : M \to M$. F is said to be a large contraction if $\varphi, \psi \in M$ with $\varphi \neq \psi$, then $d(F\varphi, F\psi) < d(\varphi, \psi)$, and if for all $\varepsilon > 0$, there exists $\eta < 1$ such that

$$[\varphi, \psi \in M, \ d(\varphi, \psi) \geq \varepsilon] \Rightarrow d(F\varphi, F\psi) \leq \eta d(\varphi, \psi).$$

Theorem 2.3 (Burton). *Let (M,d) be a complete metric space and F be a large contraction. Suppose there is $x \in M$ and $\rho > 0$ such that $d(x, F^n x) \leq \rho$ for all $n \geq 1$. Then F has a unique fixed point in M.*

Below, we state Krasnoselskii-Burton's hybrid fixed point theorem which enables us to establish a stability result of the trivial solution of

(1). For more details on Krasnoselskii's captivating theorem we refer to Smart [23] or [8].

Theorem 2.4 (Krasnoselskii-Burton). *Let M be a closed bounded convex nonempty subset of a Banach space $(S, \|.\|)$. Suppose that $\mathcal{A}$, $\mathcal{B}$ map M into M and that*

(i) for all $x, y \in M \Rightarrow \mathcal{A}x + \mathcal{B}y \in M$,

(ii) $\mathcal{A}$ is continuous and $\mathcal{A}M$ is contained in a compact subset of M,

(iii) $\mathcal{B}$ is a large contraction.

Then there is $z \in M$ with $z = \mathcal{A}z + \mathcal{B}z$.

Here we manipulate function spaces defined on infinite t-intervals. So for compactness, we need an extension of Arzela-Ascoli theorem. This extension is taken from [[8], Theorem 1.2.2, p. 20] and is as follows.

Theorem 2.5. *Let $q : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that $q(t) \to 0$ as $t \to \infty$. If $\{\varphi_n(t)\}$ is an equicontinuous sequence of $\mathbb{R}^m$-valued functions on $\mathbb{R}_+$ with $|\varphi_n(t)| \le q(t)$ for $t \in \mathbb{R}_+$, then there is a subsequence that converges uniformly on $\mathbb{R}_+$ to a continuous function $\varphi(t)$ with $|\varphi(t)| \le q(t)$ for $t \in \mathbb{R}_+$, where $|.|$ denotes the Euclidean norm on $\mathbb{R}^m$.*

3. Stability by Krasnoselskii-Burton's theorem

From the existence theory which can be found in [8], we conclude that for each continuous initial function $\phi : [m_0, t_0] \to \mathbb{R}$, there exists a continuous solution $x(t, t_0, \phi)$ which satisfies (1) on an interval $[0, \sigma)$ for some $\sigma > 0$ and $x(t, t_0, \phi) = \phi(t)$ for $t \in [m_0, t_0]$.

We need the following stability definitions taken from [8].

Definition 3.1. The zero solution of (1) is said to be stable at $t = t_0$ if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\phi : [m_0, t_0] \to (-\delta, \delta)$ implies that $|x(t, t_0, \phi)| < \varepsilon$ for all $t \ge m_0$.

Definition 3.2. The zero solution of (1) is said to be asymptotically stable if it is stable at $t = t_0$ and $\delta > 0$ exists such that for any contunuos function $\phi : [m_0, t_0] \to (-\delta, \delta)$ the solution $x(t, t_0, \phi)$ with $x(t, t_0, \phi) = \phi(t)$ on $[m_0, t_0]$ tends to zero as $t \to \infty$.

To apply Theorem 2.4, we have to choose carefully a Banach space depending on the initial function ϕ and construct two mappings, a large contraction and a compact operator which obey the conditions of the theorem. So let S be the Banach space of continuous bounded functions $\varphi : [m_0, \infty] \to \mathbb{R}$ with the supremum norm $\|.\|$. Let $L > 0$ and define the set

$$\begin{aligned} S_\phi &= \{\varphi \in S : \varphi \text{ is Lipschitzian, } |\varphi(t)| \leq L,\ t \in [m_0, \infty), \\ &\quad \varphi(t) = \phi(t) \text{ if } t \in [m_0, t_0] \text{ and } \varphi(t) \to 0 \text{ as } t \to \infty\}. \end{aligned}$$

Clearly, if $\{\varphi_n\}$ is a sequence of k-Lipschitzian functions converging to a function φ then

$$\begin{aligned} |\varphi(u) - \varphi(v)| &\leq |\varphi(u) - \varphi_n(u)| + |\varphi_n(u) - \varphi_n(v)| + |\varphi_n(v) - \varphi(v)| \\ &\leq \|\varphi - \varphi_n\| + k\,|u - v| + \|\varphi - \varphi_n\|. \end{aligned}$$

Consequently, as $n \to \infty$, we see that φ is k-Lipschitzian. It is clear that S_ϕ is convex, bounded and complete endowed with $\|.\|$.

For $\varphi \in S_\phi$ and $t \geq t_0$, define the maps $\mathcal{A}$, $\mathcal{B}$ and H on S_ϕ as follows

$$\begin{aligned} (\mathcal{A}\varphi)(t) &= -\gamma(t)\varphi(t - \tau(t)) - \int_{t_0}^{t} L_\varphi(s) e^{-\int_s^t A(z)dz} ds \\ &\quad + \int_{t_0}^{t} \mu(s)\varphi(s - \tau(s)) e^{-\int_s^t A(z)dz} ds, \end{aligned} \tag{12}$$

$$\begin{aligned} (\mathcal{B}\varphi)(t) &= (\phi(t_0) + \gamma(t_0)\phi(t_0 - \tau(t_0)))\, e^{-\int_{t_0}^t A(z)dz} \\ &\quad + \int_{t_0}^{t} \left(\int_{s-\tau(s)}^{s} a(s,u)\,(G\varphi)\,(u)du \right) e^{-\int_s^t A(z)dz} ds, \end{aligned} \tag{13}$$

and

$$(H\varphi)(t) = (\mathcal{A}\varphi)(t) + (\mathcal{B}\varphi)(t). \tag{14}$$

If we are able to prove that H possesses a fixed point φ on the set S_ϕ, then $x(t, t_0, \phi) = \varphi(t)$ for $t \geq t_0$, $x(t, t_0, \phi) = \phi(t)$ on $[m_0, t_0]$, $x(t, t_0, \phi)$ satisfies (1) when its derivative exists and $x(t, t_0, \phi) \to 0$ as $t \to \infty$.

Let

$$\omega(t) = \int_{t-\tau(t)}^{t} |a(t,s)| \left(\int_s^t \left(\int_{u-\tau(u)}^{u} |a(u,v)|\, dv + |r(u)| \right) du + |\gamma(t)| + |\gamma(s)| \right) ds,$$

and assume that there are constants $k_1, k_2, k_3 > 0$ such that for $t_0 \leq t_1 \leq t_2$,

$$\left| \int_{t_1}^{t_2} A(z)dz \right| \leq k_1 \,|t_2 - t_1|\,, \tag{15}$$

$$|\tau(t_2) - \tau(t_1)| \le k_2 |t_2 - t_1|, \tag{16}$$

and

$$|\gamma(t_2) - \gamma(t_1)| \le k_3 |t_2 - t_1|. \tag{17}$$

Suppose for $t \ge t_0$,

$$|\mu(t)| \le \delta A(t), \tag{18}$$

$$\omega(t) \le \lambda A(t), \tag{19}$$

$$\sup_{t \ge t_0} |\gamma(t)| = \alpha_0, \tag{20}$$

and that

$$J(\alpha_0 + \lambda + \delta) < 1, \tag{21}$$

$$\max(|G(-L)|, |G(L)|) \le \frac{2L}{J}, \tag{22}$$

where α_0, δ, λ, J are positive constants with $J > 3$.

Choose $\rho > 0$ small enough and such that

$$(1 + \gamma(t_0))\rho + \frac{3L}{J} \le L. \tag{23}$$

The chosen ρ in the relation (23) is used below in Lemma 3.5 to show that if $\varepsilon = L$ and if $\|\phi\| < \rho$, then the solutions satisfy $x(t, t_0, \phi) < \varepsilon$.

Assume further that

$$t - \tau(t) \to \infty \text{ as } t \to \infty \text{ and } \int_0^t A(z)dz \to \infty \text{ as } t \to \infty, \tag{24}$$

$$\gamma(t) \to 0 \text{ as } t \to \infty, \tag{25}$$

$$\frac{\mu(t)}{A(t)} \to 0 \text{ as } t \to \infty, \tag{26}$$

and

$$\frac{\omega(t)}{A(t)} \to 0 \text{ as } t \to \infty. \tag{27}$$

We begin by showing that G given by (6) is a large contraction on the set S_ϕ. So, we suppose that $g : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions.

$(H1)$ $g : \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,

$(H2)$ the function g is strictly increasing on $[-L, L]$,
$(H3)$ $\sup_{t\in(-L,L)} g'(t) \leq 1$.

THEOREM 3.3 ([1]). *Let $g : \mathbb{R} \to \mathbb{R}$ be a function satisfying $(H1) - (H3)$. Then the mapping G in (6) is a large contraction on the set S_ϕ.*

By step we will prove the fulfillment of (i), (ii) and (iii) in Theorem 2.4.

LEMMA 3.4. *Suppose that (18)–(21) and (24) hold. For $\mathcal{A}$ defined in (12), if $\varphi \in S_\phi$, then $|(\mathcal{A}\varphi)(t)| \leq L/J \leq L$. Moreover, $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$.*

Proof. Using the conditions (18)–(21) and the expression (12) of the map $\mathcal{A}$, we get

$$\begin{aligned} |(\mathcal{A}\varphi)(t)| &\leq |\gamma(t)|\,|\varphi(t-\tau(t))| + \int_{t_0}^{t} |L_\varphi(s)|\, e^{-\int_s^t A(z)dz} ds \\ &\quad + \int_{t_0}^{t} |\mu(s)|\,|\varphi(s-\tau(s))|\, e^{-\int_s^t A(z)dz} ds \\ &\leq \alpha_0 L + L\int_{t_0}^{t} \omega(s) e^{-\int_s^t A(z)dz} ds + L\int_{t_0}^{t} |\mu(s)|\, e^{-\int_s^t A(z)dz} ds \\ &\leq \alpha_0 L + \lambda L\int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds + \delta L\int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds \\ &\leq (\alpha_0+\lambda+\delta)L \leq \frac{L}{J} < L. \end{aligned}$$

So AS_ϕ is bounded by L as required.

Let $\varphi \in S_\phi$ be fixed. We will prove that $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$. Due to the conditions $t-\tau(t) \to \infty$ as $t \to \infty$ in (24) and (20), it is obvious that the first term on the right hand side of $\mathcal{A}$ tends to 0 as $t \to \infty$. That is

$$|\gamma(t)\varphi(t-\tau(t))| \leq \alpha_0\, |\varphi(t-\tau(t))| \to 0 \text{ as } t \to \infty.$$

It is left to show that the two remaining integral terms of $\mathcal{A}$ go to zero as $t \to \infty$. Let $\varepsilon > 0$ be given. Find T such that $|\varphi(t-\tau(t))| < \varepsilon$ for

$t \geq T$. Then we have

$$\begin{aligned}
&\left| \int_{t_0}^{t} L_{\varphi}(s) e^{-\int_s^t A(z)dz} ds \right| \\
&\leq \int_{t_0}^{T} |L_{\varphi}(s)| \, e^{-\int_s^t A(z)dz} ds + \int_{T}^{t} |L_{\varphi}(s)| \, e^{-\int_s^t A(z)dz} ds \\
&\leq L e^{-\int_T^t A(z)dz} \int_{t_0}^{T} \omega(s) e^{-\int_s^T A(z)dz} ds + \varepsilon \int_{T}^{t} \omega(s) e^{-\int_s^t A(z)dz} ds \\
&\leq L\lambda e^{-\int_T^t A(z)dz} + \varepsilon\lambda,
\end{aligned}$$

and

$$\begin{aligned}
&\left| \int_{t_0}^{t} \mu(s) \varphi(s - \tau(s)) e^{-\int_s^t A(z)dz} ds \right| \\
&\leq \int_{t_0}^{T} |\mu(s)| \, |\varphi(s - \tau(s))| \, e^{-\int_s^t A(z)dz} ds \\
&+ \int_{T}^{t} |\mu(s)| \, |\varphi(s - \tau(s))| \, e^{-\int_s^t A(z)dz} ds \\
&\leq L e^{-\int_T^t A(z)dz} \int_{t_0}^{T} |\mu(s)| \, e^{-\int_s^T A(z)dz} ds + \varepsilon \int_{T}^{t} |\mu(s)| \, e^{-\int_s^t A(z)dz} ds \\
&\leq L\delta e^{-\int_T^t A(z)dz} + \varepsilon\delta.
\end{aligned}$$

The terms $L\lambda e^{-\int_T^t A(z)dz}$ and $L\delta e^{-\int_T^t A(z)dz}$ are arbitrarily smalls as $t \to \infty$, because of (24). This ends the proof. □

Lemma 3.5. *Let (18)–(22) and (24) hold. For $\mathcal{A}$, $\mathcal{B}$ defined in (12) and (13), if $\varphi, \psi \in S_\phi$ are arbitrary, then*

$$\|\mathcal{A}\varphi + \mathcal{B}\psi\| \leq L.$$

Moreover, $\mathcal{B}$ is a large contraction on S_ϕ with a unique fixed point in S_ϕ and $(\mathcal{B}\psi)(t) \to 0$ as $t \to \infty$.

Proof. Using the definitions (12), (13) of $\mathcal{A}$ and $\mathcal{B}$ and applying (18)–(22), we obtain

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t)+(\mathcal{B}\psi)(t)\right| \\
&\leq \left|(\mathcal{A}\varphi)(t)\right|+\left|(\mathcal{B}\psi)(t)\right| \\
&\leq \alpha_0 L+\lambda L\int_{t_0}^{t} A(s)e^{-\int_s^t A(z)dz}ds+L\int_{t_0}^{t}\left|\mu(s)\right|e^{-\int_s^t A(z)dz}ds \\
&+(1+\gamma(t_0))\left\|\phi\right\|e^{-\int_{t_0}^t A(z)dz}+\frac{2L}{J}\int_{t_0}^{t} A(s)e^{-\int_s^t A(z)dz}ds \\
&\leq (1+\gamma(t_0))\left\|\phi\right\|+(\alpha_0+\lambda+\delta)L+\frac{2L}{J} \\
&\leq (1+\gamma(t_0))\left\|\phi\right\|+\frac{L}{J}+\frac{2L}{J},
\end{aligned}$$

by the monotonicity of the mapping G. So from the above inequality, by choosing the initial function ϕ having small norm, say $\|\phi\| \leq \rho$, then, and referring to (23), we obtain

$$\|\mathcal{A}\varphi+\mathcal{B}\psi\| \leq (1+\gamma(t_0))\rho+\frac{3L}{J}\leq L.$$

Since $0 \in S_\phi$, we have also proved that $|(\mathcal{B}\psi)(t)| \leq L$. The proof that $\mathcal{B}\psi$ is Lipschitzian is similar to that of the map $\mathcal{A}\varphi$ below. To see that $\mathcal{B}$ is a large contraction on S_ϕ with a unique fixed point, we know from Theorem 3.3 that $G(\varphi) = \varphi - g(\varphi)$ is a large contraction within the integrand. Thus, for any ε, from the proof of that Theorem 3.3, we have found $\eta < 1$ such that

$$\begin{aligned}
&\left|(\mathcal{B}\varphi)(t)-(\mathcal{B}\psi)(t)\right| \\
&\leq \int_{t_0}^{t}\left(\int_{s-\tau(s)}^{s}\left|a(s,u)\right|\left|(G\varphi)(u)-(G\psi)(u)\right|du\right)e^{-\int_s^t A(z)dz}ds \\
&\leq \eta\int_{t_0}^{t}\left(\int_{s-\tau(s)}^{s}a(s,u)\left\|\varphi-\psi\right\|du\right)e^{-\int_s^t A(z)dz}ds \\
&\leq \eta\int_{t_0}^{t}A(s)\left\|\varphi-\psi\right\|e^{-\int_s^t A(z)dz}ds \\
&\leq \eta\left\|\varphi-\psi\right\|.
\end{aligned}$$

To prove that $(\mathcal{B}\psi)(t) \to 0$ as $t \to \infty$, we use (24) for the first term, and for the second term, we argue as above for the map $\mathcal{A}$. □

LEMMA 3.6. *Suppose (18)–(21) hold. Then the mapping $\mathcal{A}$ is continuous on S_ϕ.*

Proof. Let $\varphi, \psi \in S_\phi$, then

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)| \\
&\leq \alpha_0 |\varphi(t-\tau(t)) - \psi(t-\tau(t))| + \int_{t_0}^{t} |L_\varphi(s) - L_\psi(s)| e^{-\int_s^t A(z)dz} ds \\
&+ \int_{t_0}^{t} |\mu(s)| |\varphi(s-\tau(s)) - \psi(s-\tau(s))| e^{-\int_s^t A(z)dz} ds \\
&\leq \alpha_0 \|\varphi - \psi\| + \|\varphi - \psi\| \int_{t_0}^{t} \omega(s) e^{-\int_s^t A(z)dz} ds \\
&+ \|\varphi - \psi\| \int_{t_0}^{t} |\mu(s)| e^{-\int_s^t A(z)dz} ds \\
&\leq \alpha_0 \|\varphi - \psi\| + \lambda \|\varphi - \psi\| \int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds \\
&+ \delta \|\varphi - \psi\| \int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds \\
&\leq (\alpha_0 + \lambda + \delta) \|\varphi - \psi\| \leq \frac{1}{J} \|\varphi - \psi\|.
\end{aligned}$$

Let $\varepsilon > 0$ be arbitrary. Define $\eta = \varepsilon J$. Then for $\|\varphi - \psi\| \leq \eta$, we obtain

$$\|\mathcal{A}\varphi - \mathcal{A}\psi\| \leq \frac{1}{J} \|\varphi - \psi\| \leq \varepsilon.$$

Therefore, $\mathcal{A}$ is continuous. □

LEMMA 3.7. *Let (15)–(20) and (25)–(27) hold. The function $\mathcal{A}\varphi$ is Lipschitzian and the operator $\mathcal{A}$ maps S_ϕ into a compact subset of S_ϕ.*

Proof. Let $\varphi \in S_\phi$ and let $0 \leq t_1 < t_2$. Then

(28)

$$\begin{aligned}
&|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| \\
&\leq |\gamma(t_2)\varphi(t_2 - \tau(t_2)) - \gamma(t_1)\varphi(t_1 - \tau(t_1))| \\
&+ \left| \int_{t_0}^{t_2} L_\varphi(s) e^{-\int_s^{t_2} A(z)dz} ds - \int_{t_0}^{t_1} L_\varphi(s) e^{-\int_s^{t_1} A(z)dz} ds \right| \\
&+ \left| \int_{t_0}^{t_2} \mu(s)\varphi(s-\tau(s)) e^{-\int_s^{t_2} A(z)dz} ds - \int_{t_0}^{t_1} \mu(s)\varphi(s-\tau(s)) e^{-\int_s^{t_1} A(z)dz} ds \right|.
\end{aligned}$$

By hypotheses (16)–(17), we have

(29)

$$\begin{aligned}
&|\gamma(t_2)\varphi(t_2-\tau(t_2))-\gamma(t_1)\varphi(t_1-\tau(t_1))| \\
&\le |\gamma(t_2)|\,|\varphi(t_2-\tau(t_2))-\varphi(t_1-\tau(t_1))| + |\varphi(t_1-\tau(t_1))|\,|\gamma(t_2)-\gamma(t_1)| \\
&\le \alpha_0 k\,|(t_2-t_1)-(\tau(t_2)-\tau(t_1))| + Lk_3\,|t_2-t_1| \\
&\le (\alpha_0 k+\alpha_0 k k_2 + Lk_3)\,|t_2-t_1|,
\end{aligned}$$

where k is the Lipschitz constant of φ. By hypotheses (15) and (18), we have

(30)

$$\begin{aligned}
&\left|\int_{t_0}^{t_2}\mu(s)\varphi(s-\tau(s))e^{-\int_s^{t_2}A(z)dz}ds-\int_{t_0}^{t_1}\mu(s)\varphi(s-\tau(s))e^{-\int_s^{t_1}A(z)dz}ds\right| \\
&\le \left|\int_{t_0}^{t_1}\mu(s)\varphi(s-\tau(s))e^{-\int_s^{t_1}A(z)dz}\left(e^{-\int_{t_1}^{t_2}A(z)dz}-1\right)ds\right. \\
&\left.+\int_{t_1}^{t_2}\mu(s)\varphi(s-\tau(s))e^{-\int_s^{t_2}A(z)dz}ds\right| \\
&\le L\left|e^{-\int_{t_1}^{t_2}A(z)dz}-1\right|\int_{t_0}^{t_1}\delta A(s)e^{-\int_s^{t_1}A(z)dz}ds+L\int_{t_1}^{t_2}|\mu(s)|\,e^{-\int_s^{t_2}A(z)dz}ds \\
&\le L\delta\int_{t_1}^{t_2}A(s)ds+L\int_{t_1}^{t_2}e^{-\int_s^{t_2}A(z)dz}d\left(\int_{t_1}^{s}|\mu(v)|\,dv\right) \\
&\le L\delta\int_{t_1}^{t_2}A(s)ds+L\left\{\left[e^{-\int_s^{t_2}A(z)dz}\int_{t_1}^{s}|\mu(v)|\,dv\right]_{t_1}^{t_2}\right. \\
&\left.+\int_{t_1}^{t_2}A(s)e^{-\int_s^{t_2}A(z)dz}\int_{t_1}^{s}|\mu(v)|\,dvds\right\} \\
&\le L\delta\int_{t_1}^{t_2}A(s)ds+L\int_{t_1}^{t_2}|\mu(v)|\,dv\left(1+\int_{t_1}^{t_2}A(s)e^{-\int_s^{t_2}A(z)dz}ds\right) \\
&\le L\delta\int_{t_1}^{t_2}A(s)ds+2L\int_{t_1}^{t_2}|\mu(v)|\,dv \\
&\le L\delta\int_{t_1}^{t_2}A(s)ds+2L\delta\int_{t_1}^{t_2}A(v)dv \\
&\le 3L\delta k_1\,|t_2-t_1|.
\end{aligned}$$

Similarly, by (15) and (19), we deduce

(31)

$$\begin{aligned}
&\left|\int_{t_0}^{t_2} L_\varphi(s)e^{-\int_s^{t_2} A(z)dz}ds - \int_{t_0}^{t_1} L_\varphi(s)e^{-\int_s^{t_1} A(z)dz}ds\right| \\
&= \left|\int_{t_0}^{t_1} L_\varphi(s)e^{-\int_s^{t_1} A(z)dz}\left(e^{-\int_{t_1}^{t_2} A(z)dz} - 1\right)ds + \int_{t_1}^{t_2} L_\varphi(s)e^{-\int_s^{t_2} A(z)dz}ds\right| \\
&\leq L\left|e^{-\int_{t_1}^{t_2} A(z)dz} - 1\right|\int_{t_0}^{t_1} \omega(s)e^{-\int_s^{t_1} A(z)dz}ds + L\int_{t_1}^{t_2} \omega(s)e^{-\int_s^{t_2} A(z)dz}ds \\
&\leq L\left|e^{-\int_{t_1}^{t_2} A(z)dz} - 1\right|\int_{t_0}^{t_1} \lambda A(s)e^{-\int_s^{t_1} A(z)dz}ds + L\int_{t_1}^{t_2} \omega(s)e^{-\int_s^{t_2} A(z)dz}ds \\
&\leq \lambda L\int_{t_1}^{t_2} A(z)dz + L\int_{t_1}^{t_2} e^{-\int_s^{t_2} A(z)dz}d\left(\int_{t_1}^{s} \omega(v)dv\right) \\
&\leq \lambda L\int_{t_1}^{t_2} A(z)dz + L\left\{\left[e^{-\int_s^{t_2} A(z)dz}\int_{t_1}^{s} \omega(v)dv\right]_{t_1}^{t_2}\right. \\
&\left.+\int_{t_1}^{t_2} A(s)e^{-\int_s^{t_2} A(z)dz}\int_{t_1}^{s} \omega(v)dvds\right\} \\
&\leq \lambda L\int_{t_1}^{t_2} A(z)dz + L\int_{t_1}^{t_2} \omega(v)dv\left(1 + \int_{t_1}^{t_2} A(s)e^{-\int_s^{t_2} A(z)dz}ds\right) \\
&\leq \lambda L\int_{t_1}^{t_2} A(z)dz + 2L\int_{t_1}^{t_2} \omega(v)dv \\
&\leq \lambda L\int_{t_1}^{t_2} A(z)dz + 2L\lambda\int_{t_1}^{t_2} A(v)dv \\
&\leq 3\lambda L k_1 \left|t_2 - t_1\right|.
\end{aligned}$$

Thus, by substituting (29)–(31) in (28), we obtain

(32)

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)\right| \\
&\leq (\alpha_0 k + \alpha_0 k k_2 + L k_3)\left|t_2 - t_1\right| + 3L\delta k_1\left|t_2 - t_1\right| + 3L\lambda k_1\left|t_2 - t_1\right| \\
&\leq K\left|t_2 - t_1\right|,
\end{aligned}$$

for a constant $K > 0$. This shows $\mathcal{A}\varphi$ that is Lipschitzian if φ is and that $\mathcal{A}S_\phi$ is equicontinuous. Next, we notice that for arbitrary $\varphi \in S_\phi$, we have

$$
\begin{aligned}
&|(\mathcal{A}\varphi)(t)| \\
&\leq |\gamma(t)\varphi(t-\tau(t))| + \int_{t_0}^{t} |L_\varphi(s)|\, e^{-\int_s^t A(z)dz} ds \\
&+ \int_{t_0}^{t} |\mu(s)|\,|\varphi(s-\tau(s))|\, e^{-\int_s^t A(z)dz} ds \\
&\leq L\,|\gamma(t)| + L\int_{t_0}^{t} \omega(s) e^{-\int_s^t A(z)dz} ds + L\int_{t_0}^{t} |\mu(s)|\, e^{-\int_s^t A(z)dz} ds \\
&\leq L\,|\gamma(t)| + L\int_{t_0}^{t} A(s)\frac{\omega(s)}{A(s)} e^{-\int_s^t A(z)dz} ds + L\int_{t_0}^{t} A(s)\frac{|\mu(s)|}{A(s)} e^{-\int_s^t A(z)dz} ds \\
&:= q(t),
\end{aligned}
$$

because of (25)–(27). Using a method like the one used for the map $\mathcal{A}$, we see that $q(t) \to 0$ as $t \to \infty$. By Theorem 2.5, we conclude that the set $\mathcal{A}S_\phi$ resides in a compact set. □

THEOREM 3.8. *Let $L > 0$. Suppose that the conditions $(H1) - (H3)$, (2) and (25)–(27) hold. If ϕ is a given initial function which is sufficiently small, then there is a solution $x(t, t_0, \phi)$ of (1) with $|x(t, t_0, \phi)| \leq L$ and $x(t, t_0, \phi) \to 0$ as $t \to \infty$.*

Proof. From Lemmas 3.4 and 3.7 we have $\mathcal{A}$ is bounded by L, Lipschitzian and $(\mathcal{A}\varphi)(t) \to 0$ as $t \to \infty$. So $\mathcal{A}$ maps S_ϕ into S_ϕ. From Lemmas 3.5 and 3.7 for arbitrary, we have $\varphi, \psi \in S_\phi$, $\mathcal{A}\varphi + \mathcal{B}\psi$ since both $\mathcal{A}\varphi$ and $\mathcal{B}\psi$ are Lipschitzian bounded by L and $(\mathcal{B}\psi)(t) \to 0$ as $t \to \infty$. From Lemmas 3.6 and 3.7, we have proved that $\mathcal{A}$ is continuous and $\mathcal{A}S_\phi$ resides in a compact set. Thus, all the conditions of Theorem 2.4 are satisfied. Therefore, there exists a solution of (1) with $|x(t, t_0, \phi)| \leq L$ and $x(t, t_0, \phi) \to 0$ as $t \to \infty$. □

4. Stability in weighted Banach spaces

Referring to Burton [8], except for the fixed point method, we know of no other way proving that solutions of (1) converge to zero. Nevertheless, if all we need is stability and not asymptotic stability, then we can avoid conditions (25)–(27) and still use Krasnoselskii-Burton's theorem on a Banach space endowed with a weighted norm.

Let $h : [m_0, \infty) \to [1, \infty)$ be any strictly increasing and continuous function with $h(m_0) = 1$, $h(s) \to \infty$ as $s \to \infty$. Let $(S, |.|_h)$ be the Banach space of continuous $\varphi : [m_0, \infty) \to \mathbb{R}$ for which

$$|\varphi|_h = \sup_{t \geq m_0} \left| \frac{\varphi(t)}{h(t)} \right| < \infty,$$

exists. We continue to use $\|.\|$ as the supremum norm of any $\varphi \in S$ provided φ bounded. Also, we use $\|\phi\|$ as the bound of the initial function. Further, in a similar way as Theorem 3.3, we can prove that the function $G(\varphi) = \varphi - g(\varphi)$ is still a large contraction with the norm $|.|_h$.

THEOREM 4.1. *If the conditions of Theorem 3.8 hold, except for (25)–(27), then the zero solution of (1) is stable.*

Proof. We prove the stability starting at t_0. Let $\varepsilon > 0$ be given such that $0 < \varepsilon < L$, then for $|x| \leq \varepsilon$, find α^* with $|x - g(x)| \leq \alpha^*$ and choose a number α such that

$$\alpha + \alpha^* + \frac{\varepsilon}{J} \leq \varepsilon. \tag{33}$$

In fact, since $x - g(x)$ is increasing on $(-L, L)$, we may take $\alpha^* = \frac{2\varepsilon}{J}$. Thus, inequality (33) allows $\alpha > 0$. Now, remove the condition $\varphi(t) \to 0$ as $t \to \infty$ from S_ϕ defined previously and consider the set

$$\begin{aligned} E_\phi \;=\; & \{\varphi \in S : \varphi \text{ Lipshitzian, } |\varphi(t)| \leq \varepsilon,\, t \in [m_0, \infty) \\ & \text{and } \varphi(t) = \phi(t) \text{ for } t \in [m_0, t_0]\}. \end{aligned}$$

Define $\mathcal{A}$, $\mathcal{B}$ on E_ϕ as before by (12), (13). We easily check that if $\varphi \in E_\phi$, then $|(\mathcal{A}\varphi)(t)| \leq \varepsilon$, and $\mathcal{B}$ is a large contraction on E_ϕ. Also, by choosing $\|\phi\| \leq \alpha$ and referring to (33), we verify that for $\varphi, \psi \in E_\phi$, $|(\mathcal{A}\varphi)(t) + (\mathcal{B}\psi)(t)| \leq \varepsilon$ and $|(\mathcal{B}\psi)(t)| \leq \varepsilon$. $\mathcal{A}E_\phi$ is an equicontinuous set. According to [[8], Theorem 4.0.1], in the space $(S, |.|_h)$ the set $\mathcal{A}E_\phi$ resides in a compact subset of E_ϕ. Moreover, the operator $\mathcal{A} : E_\phi \to E_\phi$

is countinuous. Indeed, for $\varphi, \psi \in S_\phi$,

$$\begin{aligned}
&\frac{|(\mathcal{A}\varphi)(t) - (\mathcal{A}\psi)(t)|}{h(t)} \\
&\leq \frac{1}{h(t)} \{|\gamma(t)| |\varphi(t-\tau(t)) - \psi(t-\tau(t))| \\
&+ \left| \int_{t_0}^{t} (L_\varphi(s) - L_\psi(s)) e^{-\int_s^t A(z)dz} ds \right| \\
&+ \left| \int_{t_0}^{t} \mu(s) (\varphi(s-\tau(s)) - \psi(s-\tau(s))) e^{-\int_s^t A(z)dz} ds \right| \Big\} \\
&\leq \alpha_0 |\varphi - \psi|_h + |\varphi - \psi|_h \int_{t_0}^{t} \omega(s) \frac{h(s)}{h(t)} e^{-\int_s^t A(z)dz} ds \\
&+ |\varphi - \psi|_h \int_{t_0}^{t} |\mu(s)| \frac{h(s-\tau(s))}{h(t)} e^{-\int_s^t A(z)dz} ds \\
&\leq \alpha_0 |\varphi - \psi|_h + \lambda |\varphi - \psi|_h \int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds \\
&+ \delta |\varphi - \psi|_h \int_{t_0}^{t} A(s) e^{-\int_s^t A(z)dz} ds \\
&\leq (\alpha_0 + \lambda + \delta) |\varphi - \psi|_h \leq \frac{1}{J} |\varphi - \psi|_h .
\end{aligned}$$

The conditions of Theorem 2.4 are satisfied on E_ϕ, and so there exists a fixed point lying in E_ϕ and solving (1). □

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