Korean J. Math. **25** (2017), No. 2, pp. 211–214
https://doi.org/10.11568/kjm.2017.25.2.211

A REMARK ON A TRIPLE POINTS IN THE BOUNDARY OF QUATERNIONIC HYPERBOLIC SPACE

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Abstract. In this paper we consider a triple of distinct points in the boundary of quaternionic hyperbolic space and detect where these points are by using the quaternionic triple product.

1. Introduction

When a triple of distinct points are given on the boundary of quaternionic hyperbolic space, by using the Cartan angular invariant, one can determine whether these three points lie in a same $\mathbb{R}$-circle or in the boundary of $\mathbb{H}$-line.(See [1]) More precisely, B.Apanasov and I.Kim proved the following theorem.

Theorem 1.1. (Theorem 3.5 and 3.6 in [1]) *A triple* $x = (x_1, x_2, x_3) \in (\partial \mathbf{H}^{\mathbf{n}}_{\mathbb{H}})^{\mathbf{3}}$ *lies in the boundary of an* $\mathbb{H}$*-line if and only if* $\mathbb{A}_{\mathbb{H}}(p) = \pi/2$, *and lies in the same* $\mathbb{R}$*-circle if and only if* $\mathbb{A}_{\mathbb{H}}(p) = 0$.

Here $\mathbb{A}_{\mathbb{H}}(p) = \pi/2$ if and only if $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ is purely imaginary and $\mathbb{A}_{\mathbb{H}}(p) = 0$ if and only if $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \in \mathbb{R}$ respectively.(We will define the Cartan angular invariant $\mathbb{A}_{\mathbb{H}}(p)$ and the triple $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ in next chapter.)
In this article, we give answer to the question that where these three

Received April 11, 2017. Revised May 17, 2017. Accepted May 24, 2017.
2010 Mathematics Subject Classification: 20H10, 30F35, 30F40, 57S30.
Key words and phrases: Quaternionic hyperbolic space, Quaternionic Cartan angular invariant.
This work was supported by 2016 Hannam University Research Fund.

points are when the triple $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ is other values such as a complex number or of the form $a + bj$ or $a + bk$ where $a, b \in \mathbb{R}$.

2. Quaternionic Cartan angular invariant

The projective model of the quaternionic hyperbolic space $H^n_{\mathbb{H}}$ is the set of negative lines in the Hermitian vector space $\mathbb{H}^{n,1}$ with Hermitian structure defined by the indefinite $(n, 1)$-form

$$\langle\langle z, w \rangle\rangle = z_1 \overline{w_1} + \cdots + z_n \overline{w_n} - z_{n+1} \overline{w_{n+1}}.$$

One can obtain the ball model $B^n_{\mathbb{H}}(0, 1) \subset \mathbb{H}^n$ by taking inhomogeneous coordinates. Here, throughout this article, we use the left vector space $\mathbb{H}^{n,1}$, in which multiplication by quaternion numbers is on the left. For more details on quaternionic hyperbolic geometry, we refer [1], [3] or [4]. The Cartan angular invariant is well-known invariant in complex hyperbolic geometry, but in quaternionic hyperbolic geometry, B.N.Apanasov and I.Kim first defined it in [1]. Here we give the definition and some properties.
Let $x = (x_1, x_2, x_3) \in (H^n_{\mathbb{H}} \cup \partial H^n_{\mathbb{H}})^3$ be a triple of distinct points with lifts $\tilde{x}_i \in H^{n,1}_{\mathbb{H}}$ for $i = 1, 2, 3$. Then the quaternionic Cartan angular invariant $\mathbb{A}_{\mathbb{H}}(x)$ of a triple $x = (x_1, x_2, x_3)$ is the angle between the first coordinate line $\mathbb{R}e_0 = (\mathbb{R}, 0, 0, 0) \subset \mathbb{R}^4 \cong \mathbb{H}$ and the radius vector of the quaternion equal to the Hermitian triple product $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = \langle \tilde{x}_1, \tilde{x}_2 \rangle \langle \tilde{x}_2, \tilde{x}_3 \rangle \langle \tilde{x}_3, \tilde{x}_1 \rangle \in \mathbb{H}$. We list some properties of this invariant. One can check them easily or find the proofs in [1].

(1) $\mathbb{A}_{\mathbb{H}}(x)$ is independent of the choice of the lifts $\tilde{x}_i$ of the x_i.
(2) $\mathbb{A}_{\mathbb{H}}(x)$ is invariant under permutations of the points x_i.
(3) For $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, $\mathbb{A}_{\mathbb{H}}(x) = \mathbb{A}_{\mathbb{H}}(y)$ if and only if there exists an isometry $f \in \mathbf{PSp}(n, 1)$ of $H^n_{\mathbb{H}}$ such that $f(x_i) = y_i$ for $i = 1, 2, 3$.

In addition, B.Apanasov and I.Kim showed the following theorems.

Theorem 2.1. (Theorem 3.5 in [1]) *A triple* $x = (x_1, x_2, x_3) \in (\partial H^n_{\mathbb{H}})^3$ *lies in the same* $\mathbb{R}$*-circle if and only if* $\mathbb{A}_{\mathbb{H}}(x) = 0$.

Theorem 2.2. (Theorem 3.6 in [1]) *A triple* $x = (x_1, x_2, x_3) \in (\partial H^n_{\mathbb{H}})^3$ *lies in the boundary of an* $\mathbb{H}$*-line if and only if* $\mathbb{A}_{\mathbb{H}}(x) = \pi/2$.

REMARK 2.3. In the above theorems, $\mathbb{A}_{\mathbb{H}}(x) = 0$ means that $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \in \mathbb{R}$ and $\mathbb{A}_{\mathbb{H}}(x) = \pi/2$ means that $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ is purely imaginary.

3. Main theorem

From now on, we will focus on the triple product $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ instead of $\mathbb{A}_{\mathbb{H}}$.

THEOREM 3.1. *Let* $K = \{(q_1, q_2) \in H^2_{\mathbb{H}} | q_1 \in \mathbb{H}, q_2 \in \mathbb{C}\}$. *Then a triple points* $x_1, x_2, x_3 \in \partial H^2_{\mathbb{H}}$ *lies in a copy of* K *if and only if the triple product* $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \in \mathbb{C}$.

Proof. First, assume that $x_1, x_2, x_3 \in \partial H^2_{\mathbb{H}}$ lies in a copy of K. Without loss of generality, we may assume that $x_1 = (0, -1), x_2 = (0, 1), x_3 = (q_1, q_2)$, where $q_1 \in \mathbb{H}$, $q_2 \in \mathbb{C}$ and $|q_1|^2 + |q_2|^2 = 1$. Then $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = -2(\overline{q}_2 - 1)(-q_2 - 1) = -2\{|q_1|^2 + (q_2 - \overline{q}_2)\} \in \mathbb{C}$.
Conversely, up to isometry, we can assume that $x_1 = (0, -1), x_2 = (0, 1), x_3 = (q_1, q_2)$ for q_1, q_2 are quaternions, $|q_1|^2 + |q_2|^2 = 1$ and $\tilde{x}_1 = (0, -1, 1), \tilde{x}_2 = (0, 1, 1), \tilde{x}_3 = (q_1, q_2, 1)$. Then $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = -2(|q_1|^2 + 2\mathrm{Im}(q_2)) \in \mathbb{C}$, so $q_2 \in \mathbb{C}$.

□

REMARK 3.2. In the above theorem, when we replace the condition $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \in \mathbb{C}$ with $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle$ is of the form $a+bj$ or $a+bk$ for $a, b \in \mathbb{R}$, one can easily checked that K is replaced with $K' = \{(q_1, q_2) \in H^2_{\mathbb{H}} | q_1 \in \mathbb{H}, q_2$ is of the form $a + bj \}$ or $K'' = \{(q_1, q_2) \in H^2_{\mathbb{H}} | q_1 \in \mathbb{H}, q_2$ is of the form $a + bk \}$

REMARK 3.3. In the theorem, the set K is similar to the bisector in the complex hyperbolic space.(See [2])

The following theorem is a special case of Theorem 2.2 and also a special case of the above theorem. By the way, it is also analogous of the result in complex hyperbolic Cartan angular invariant.

THEOREM 3.4. *A triple* $x = (x_1, x_2, x_3) \in (\partial H^2_{\mathbb{H}})^3$ *lies in a copy of* $H^1_{\mathbb{C}}$ *if and only if the triple product* $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle \in \mathbb{R}i$.

Proof. First, assume that a triple $x = (x_1, x_2, x_3) \in (\partial H^2_{\mathbb{H}})^3$ lies in a copy of $H^1_{\mathbb{C}}$. Without loss of generality, we may assume that $x_1, x_2, x_3 \in \partial H^1_{\mathbb{C}}$ and $x_1 = (0, -1), x_2 = (0, 1), x_3 = (0, z)$, where $|z| = 1$, $z \in \mathbb{C}$. Then $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = -2(z - \overline{z}) \in \mathbb{R}i$.

Conversely, up to isometry, we can assume that $x_1 = (0, -1), x_2 = (0, 1), x_3 = (q_1, q_2)$ for q_1, q_2 are quaternions, $|q_1|^2 + |q_2|^2 = 1$ and $\tilde{x}_1 = (0, -1, 1), \tilde{x}_2 = (0, 1, 1), \tilde{x}_3 = (q_1, q_2, 1)$. Then $\langle \tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \rangle = -2(|q_1|^2 + 2\mathrm{Im}(q_2))$, so $q_2 \in \mathbb{C}$ since $|q_1|^2 + 2\mathrm{Im}(q_2) \in \mathbb{R}i$. Hence $q_1 = 0$ and $q_2 \in \mathbb{C}$, so x_3 is also in $H^1_{\mathbb{C}}$. □

Acknowledgement. The author thanks to Inkang Kim and Sungwoon Kim for encouragements and useful discussions.

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