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# FUZZY HOMOMORPHISM THEOREMS ON GROUPS

Gezahagne Mulat Addis

Abstract. In this paper we introduce the notion of a fuzzy kernel of a fuzzy homomorphism on groups and we show that it is a fuzzy normal subgroup of the domain group. Conversely, we also prove that any fuzzy normal subgroup is a fuzzy kernel of some fuzzy epimorphism, namely the canonical fuzzy epimorphism. Finally, we formulate and prove the fuzzy version of the fundamental theorem of homomorphism and those isomorphism theorems.

## 1. Introduction

It is well known that if $f : G \to H$ is an ordinary(crisp) homomorphism of groups then its kernel is a normal subgroup of the domain group $G$. Conversely, given a normal subgroup $N$ of a group $G$, there exists an epimorphism $\pi$, namely the canonical epimorphism from $G$ to the quotient group $\frac{G}{N}$ defined as $\pi(a) = a + N$ for all $a \in G$ such that its kernel is $N$. Moreover, given a homomorphism $f : G \to H$, the quotient group $\frac{G}{ker\ f}$ is isomorphic to the image of $f$ which is a subgroup of the codomain group $H$. This isomorphism, let say $\overline{f}$ is simply induced by $f$; in the sense that $\overline{f}$ can be defined as $\overline{f}(a + ker\ f) = f(a)$ for all $a \in G$. In other words we have a decomposition $f = \overline{f} \circ \pi$. This is known as the fundamental theorem of homomorphisms. As a consequence of this result, the first, the second and the third isomorphism theorems follow

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immediately. In this paper we introduce the notion of fuzzy kernel of a fuzzy homomorphism on groups and we show that it is a fuzzy normal subgroup of the domain group. Conversely, it is proved in this paper that any fuzzy normal subgroup is a fuzzy kernel of some fuzzy epimorphism, namely the canonical fuzzy epimorphism. Finally, we formulate and prove the fuzzy version of the fundamental theorem of homomorphism and those isomorphism theorems analogously in a different way other than they appears in ( [1], [5], [6], [7] and [15]).

## 2. Preliminaries

In this section we recall some necessary definitions and results for a better completeness of the paper. We refer [4] for those ordinary(crisp) concepts in group theory.
For a set $X$, by a fuzzy subset of $X$, we mean a mapping of $X$ into the unit interval $[0,1]$. We say that a fuzzy subset $\mu$ of $X$ is nonzero if there is some $x \in X$ such that $\mu(x) \neq 0$. We denote by $0_X$ and $1_X$; fuzzy subsets of $A$ defined by:

$$0_X(x) = 0 \text{ and } 1_X(x) = 1 \text{ for all } x \in X$$

For any sets $X$ and $Y$, by a fuzzy relation from $X$ to $Y$, we mean a fuzzy subset of $X \times Y$ [14]. The following definition of fuzzy mappings is the generalization of the definitions found in [2], [3], [9].

Definition 2.1. A fuzzy relation $f$ from $X$ to $Y$ is said to be a fuzzy mapping from $X$ into $Y$ if for each $x \in X$ there exists a unique $y \in Y$ such that $f(x,y) = 1$.

A fuzzy mapping $f$ from $X$ into $Y$ is one-one if for each $x_1, x_2 \in X$ and $y \in Y$: $f(x_1,y) = 1$ and $f(x_2,y) = 1$ implies that $x_1 = x_2$. We call $f$ an onto fuzzy mapping if for each $y \in Y$ there exists $x \in X$ such that $f(x,y) = 1$. Moreover its fuzzy image denoted by $R_f$ is a fuzzy subset of $Y$ defined by

$$R_f(y) = sup\{f(x,y) : x \in X\}$$

for all $y \in Y$ [13]. It is observed that $f$ is onto if and only if $R_f = 1_Y$

Definition 2.2. [13] Let $f$ be a fuzzy mapping of $X$ into $Y$. If $\mu$ is a fuzzy subset of $X$ and $\nu$ is a fuzzy subset of $Y$ then

$$f(\mu)(y) = sup\{\mu(x) \wedge f(x,y) \mid x \in X\}$$

and

$$f^{-1}(\nu)(x) = sup\{f(x, y) \wedge \nu(y) \mid y \in Y\}$$

LEMMA 2.3. *Let $f$ be a fuzzy mapping of $X$ into $Y$, $\mu$ a fuzzy subset of $X$ and $\nu$ a fuzzy subset of $Y$ Then*

1. $\mu \subseteq f^{-1}(f(\mu))$
2. *If $f$ is onto, then* $\nu \subseteq f(f^{-1}(\nu))$

DEFINITION 2.4. [13] Let $f$ be a fuzzy mapping of $X$ into $Y$ and $g$ be a fuzzy mapping of $Y$ into $Z$. Then their composition $g \circ f$ defined by:

$$g \circ f(x, z) = sup\{f(x, y) \wedge g(y, z) : y \in Y\}$$

is a fuzzy mapping of $X$ into $Z$

DEFINITION 2.5. ( [10], [12]) A fuzzy relation $\Theta$ on a fuzzy equivalence relation on $X$ if the following conditions are satisfied for all $x, y, z \in X$:

1. $\Theta(x, x) = 1$
2. $\Theta(x, y) = \Theta(y, x)$
3. $\Theta(x, y) \wedge \Theta(y, z) \leq \Theta(x, z)$

Let $(G, *)$ be a group. We write $xy$ for $x * y$.

DEFINITION 2.6. [12] A fuzzy subset $\mu$ of $G$ is called a fuzzy normal subgroup of $G$ if:

1. $\mu(e) = 1$
2. $\mu(xy) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$
3. $\mu(x^{-1}) \geq \mu(x)$ for all $x \in G$
4. $\mu(xy) = \mu(yx)$ for all $x, y \in G$

Note that according to Rosenfield [11], a fuzzy normal subgroup $\mu$ need not necessarily attain the value 1 at $e$.

DEFINITION 2.7. ( [10], [12]) A fuzzy equivalence relation $\Theta$ on $G$ is a fuzzy congruence relation on $G$ if:

$$\Theta(ac, bd) \geq \Theta(a, b) \wedge \Theta(c, d)$$

for all $a, b, c, d \in G$. We denote the set of all fuzzy normal subgroups of $G$ by $FN(G)$ and the set of all fuzzy congruences on $G$ by $FC(G)$. It is observed in [12] that there is an order isomorphism between the lattice $FN(G)$ and the lattice $FC(G)$.

For a fuzzy normal subgroup $\mu$ of $G$ and each $a \in G$ define a fuzzy subset $\mu_a$, of $G$ by $\mu_a(x) = \mu(xa^{-1})$ for all $x \in G$. $\mu_a$ is called the fuzzy coset of $G$ determined by $a$ and $\mu$ [8]. It is observed that $\mu_a = \mu_b$ if and only if $\mu(ab^{-1}) = \mu(e) = 1$. The collection $\frac{G}{\mu} = \{\mu_a : a \in G\}$ forms a group under the operations defined by: $(\mu_a)(\mu_b) = \mu_{(ab)}$ for all $a, b \in G$ and it is called the quotient group induced by the fuzzy normal subgroup $\mu$. For a fuzzy subgroup $\mu$ of $G$ we denote by $G_\mu$, the level set $G_\mu = \{x \in G : \mu(x) = 1\}$.

## 3. Fuzzy Homomorphism Theorems

Throughout this section $G$ and $H$ are groups unless and otherwise stated. The following definition is due to [13].

Definition 3.1. [13] A fuzzy mapping $f$ from $G$ to $H$ is said to be a fuzzy homomorphism if for any $x_1, x_2 \in G$ and $y \in H$ we have the following:

1. $f(x_1x_2, y) = sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y_1, y_2 \in G', y_1y_2 = y\}$
2. $f(x^{-1}, y^{-1}) = f(x, y)$

If $f$ is one-one(respectively onto) then as usual we call $f$ a fuzzy monomorphism(respectively a fuzzy epimorphism) and if it is both one-one and onto then we call it a fuzzy isomorphism. If there is a fuzzy isomorphism from $G$ onto $H$ then we say that $G$ and $H$ are $F-$isomorphic to each other and we denote this by $G \overset{F}{\cong} H$

Lemma 3.2. *Let $f$ be a fuzzy homomorphism from $G$ to $H$. Then $f(e, e') = 1$ where $e$ and $e'$ are identity elements of $G$ and $H$ respectively*

Lemma 3.3. *$f$ is a fuzzy homomorphism if and only if*

$$f(x_1x_2^{-1}, y) = sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y_1y_2^{-1} = y\}$$

*Proof.* Suppose that $f$ is a fuzzy homomorphism. For any $x_1, x_2 \in G$ and $y \in H$ consider

$$\begin{aligned} f(x_1x_2^{-1}, y) &= sup\{f(x_1, y_1) \wedge f(x_2^{-1}, y_2) : y_1, y_2 \in H, y_1y_2 = y\} \\ &= sup\{f(x_1, y_1) \wedge f(x_2, y_2^{-1}) : y_1y_2 = y\} \\ &= sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y_1y_2^{-1} = y\} \end{aligned}$$

Conversely suppose that

$$f(x_1x_2^{-1}, y) = sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y_1y_2^{-1} = y\}$$

for all $x_1, x_2 \in G$ and $y \in H$. Then

$$\begin{aligned} f(x^{-1}, y^{-1}) &= f(ex^{-1}, y^{-1}) \\ &= sup\{f(e, y_1) \wedge f(x, y_2) : y_1 y_2^{-1} = y^{-1}\} \\ &\geq f(e, y_1) \wedge f(x, y_2), \forall y_1, y_2 \in H, y_1 y_2^{-1} = y^{-1} \end{aligned}$$

In particular for $y_1 = e'$ and $y_2 = y$. That is;

$$\begin{aligned} f(x^{-1}, y^{-1}) &\geq f(e, e') \wedge f(x, y) \\ &= f(x, y) \end{aligned}$$

Similarly doing we get $f(x, y) \geq f(x^{-1}, y^{-1})$ so that $f(x^{-1}, y^{-1}) = f(x, y)$.

$$\begin{aligned} f(x_1 x_2, y) &= sup\{f(x_1, y_1) \wedge f(x_2^{-1}, y_2) : y_1 y_2^{-1} = y\} \\ &= sup\{f(x_1, y_1) \wedge f(x_2, y_2^{-1}) : y_1 y_2^{-1} = y\} \\ &= sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y_1 y_2 = y\} \end{aligned}$$

□

Lemma 3.4. [13] *If $f$ is a fuzzy homomorphism from $G$ to $H$ and $g$ is a fuzzy homomorphism from $H$ to $K$, then $g \circ f$ is a fuzzy homomorphism from $G$ to $K$.*

Lemma 3.5. [13] *Let $f$ be a fuzzy homomorphism from $G$ to $H$. If $\mu$ and $\nu$ are fuzzy subgroups of $G$ and $H$ respectively then $f(\mu)$ and $f^{-1}(\nu)$ are fuzzy subgroups of $H$ and $G$ respectively.*

Definition 3.6. [13] A fuzzy homomorphism $f$ from $G$ into $H$ is said to satisfy the surjerction property if:

$$f(x, y_1 y_2) = sup\{f(x_1, y_1) \wedge f(x_2, y_2) : x_1 x_2 = x\}$$

for all $x \in G$ and all $y_1, y_2 \in H$

Lemma 3.7. [13] *Let $f$ be a fuzzy homomorphism from $G$ to $H$.*

1. *If $\nu$ is a fuzzy normal subgroup of $H$, then $f^{-1}(\nu)$ is a fuzzy normal subgroup of $G$*
2. *If $\mu$ is a fuzzy normal subgroup of $G$ and $f$ satisfies the surjection property, then $f(\mu)$ is a fuzzy normal subgroup of $H$*

Lemma 3.8. *If $f$ is a fuzzy homomorphism from $G$ to $H$, $\mu$ and $\sigma$ are fuzzy subgroups of $G$, $\nu$ and $\theta$ are fuzzy subgroups of $H$, then we have the following:*

1. *$f(\mu\sigma) = f(\mu)f(\sigma)$*
2. *$f^{-1}(\nu)f^{-1}(\theta) \subseteq f^{-1}(\nu\theta)$; and the equality holds whenever $f$ satisfies the surjection property.*

Definition 3.9. Let $f$ be a fuzzy homomorphism from $G$ to $H$. Let $K_f$ be a fuzzy subset of $G$ defined by:

$$K_f(x) = f(x, e') \text{ for all } x \in G$$

we call $K_f$ the fuzzy kernel of $f$.

Lemma 3.10. *$K_f = f^{-1}(\chi_{\{e'\}})$; where $e'$ is an identity element of $H$.*

Lemma 3.11. *$K_f$ is a normal fuzzy subgroup of $G$.*

*Proof.* Clearly $K_f(e) = 1$. Also for each $x_1, x_2 \in G$.

$$\begin{aligned} K_f(x_1x_2^{-1}) &= f(x_1x_2^{-1}, e') \\ &= sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y_1y_2^{-1} = e'\} \\ &\geq f(x_1, y_1) \wedge f(x_2, y_2), \forall y_1, y_2 \in H, y_1y_2^{-1} = e' \\ &\geq f(x_1, y_1) \wedge f(x_2, y_2), \forall y_1, y_2 \in H, y_1 = y_2 \\ &\geq f(x_1, y) \wedge f(x_2, y), \forall y \in H \end{aligned}$$

In particular $K_f(x_1x_2^{-1}) \geq f(x_1, e') \wedge f(x_2, e') = K_f(x_1) \wedge K_f(x_2)$. Therefore $K_f$ is a fuzzy subgroup of $G$. Next we show that $K_f$ is normal. For;

$$\begin{aligned} K_f(x_1x_2) &= f(x_1x_2, e') \\ &= sup\{f(x_1, y_1) \wedge f(x_2, y_2) : y_1y_2 = e'\} \\ &= sup\{f(x_2, y_2) \wedge f(x_1, y_1) : y_2y_1 = e'\} \\ &= f(x_2x_1, e') \\ &= K_f(x_2x_1) \end{aligned}$$

Therefore $K_f$ is a fuzzy normal subgroup of $G$. □

Lemma 3.12. *A fuzzy homomorphism $f$ from $G$ to $H$ is one-one if and only if $G_{(K_f)} = \{e\}$.*

**Remark**: The fuzzy kernel of $f$ can also be defined as a fuzzy relation $\Theta_f$ on the domain group $G$ as follows:

$$\Theta_f(a, b) = Sup\{f(a, y) \wedge f(b, y) : y \in H\}$$

for all $a, b \in G$. It is observed that $\Theta_f$ is a fuzzy congruence relation on $G$ and the fuzzy normal subgroup induced by $\Theta_f$ is precisely $K_f$. For this reason the quotient group $\frac{G}{\Theta_f}$ is the same as $\frac{G}{K_f}$.

Lemma 3.13. *Every fuzzy normal subgroup of $G$ is a fuzzy kernel of some fuzzy epimorphism.*

*Proof.* Let $\mu$ be a fuzzy normal subgroup of $G$. Consider the quotient group $\frac{G}{\mu}$ and define a fuzzy relation $\pi$ from $G$ into $\frac{G}{\mu}$ by:

$$\pi(x, \mu_y) = \mu(xy^{-1}) \text{ for all } x, y \in G$$

Then $\pi$ is a fuzzy epimorphism of $G$ onto $\frac{G}{\mu}$ such that its fuzzy kernel $k_\pi$ is $\mu$. We call this $\pi$ the natural (or the canonical) fuzzy epimorphism. □

Theorem 3.14. (Fundamental Theorem of Fuzzy Homomorphisms) *Let $f$ be a fuzzy homomorphism from $G$ to $H$ and $\mu$ a normal fuzzy subgroup of $G$ such that $\mu \subseteq K_f$, then there exists a fuzzy homomorphism $\overline{f}$ from $\frac{G}{\mu}$ into $H$ such that $\overline{f} \circ \pi = f$ where $\pi$ is the natural fuzzy epimorphism of $G$ onto $\frac{G}{\mu}$ and the fuzzy kernel $K_{\overline{f}} = \frac{K_f}{\mu}$. Moreover $\overline{f}$ is a fuzzy isomorphism if and only if $f$ is a fuzzy epimorphism and $G_{(K_f)} = G_\mu$.*

The theorem can be rephrased as follows: There exists a fuzzy homomorphism $\overline{f}$ from $\frac{G}{\mu}$ into $H$ such that the diagram:

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/\mu \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

is commutative.

*Proof.* Define a fuzzy relation $\overline{f}$ of $\frac{G}{\mu}$ into $H$ by

$$\overline{f}(\mu_x, y) = f(x, y); \forall x \in G, y \in H$$

We first show that $\overline{f}$ is a fuzzy mapping; for if $x_1, x_2 \in G$,such that $\mu_{x_1} = \mu_{x_2}$ and $\overline{f}(\mu_{x_1}, y_1) = 1$, $\overline{f}(\mu_{x_2}, y_2) = 1$; that is, $f(x_1, y_1) = 1$ and $f(x_2, y_2) = 1$ which implies that $f(x_1x_2^{-1}, y_1y_2^{-1}) = 1$. Also, since $\mu_{x_1} = \mu_{x_2}$ and $\mu \subseteq K_f$, we get $K_f(x_1x_2^{-1}) = 1$. That is, $f(x_1x_2^{-1}, e') = 1$ and $f(x_1x_2^{-1}, y_1y_2^{-1}) = 1$ which implies that $y_1y_2^{-1} = e'$ so that $y_1 = y_2$. Therefore $\overline{f}$ is well defined. It can also be observed that $\overline{f}$ is a fuzzy homomorphism. Now we show that $\overline{f} \circ \pi = f$; for,
For any $x \in G$ and $z \in H$ consider;

$$\begin{aligned} \overline{f} \circ \pi(x, z) &= sup\{\pi(x, \mu_y) \wedge \overline{f}(\mu_y, z) : y \in G\} \\ &= sup\{\mu(xy^{-1}) \wedge f(y, z) : y \in G\} \\ &\geq \mu(xy^{-1}) \wedge f(y, z)\ \forall y \in G \end{aligned}$$

In particular for $y = x$; that is,

$$\overline{f} \circ \pi(x, z) \geq \mu(xx^{-1}) \wedge f(x, z) = f(x, z)$$

On the other hand for any $y \in G$ consider;

$$\begin{aligned} f(x, z) &= f(xy^{-1}y, z) \\ &= sup\{f(xy^{-1}, z_1) \wedge f(y, z_2) : z_1z_2 = z\} \\ &\geq f(xy^{-1}, z_1) \wedge f(y, z_2); \forall z_1z_2 = z \end{aligned}$$

In particular for $z_1 = e'$ and $z_2 = z$; that is,

$$\begin{aligned} f(x, z) &\geq f(xy^{-1}, e') \wedge f(y, z) \\ &= K_f(xy^{-1}) \wedge \overline{f}(\mu_y, z) \\ &\geq \mu(xy^{-1}) \wedge \overline{f}(\mu_y, z) \\ &\geq \pi(x, \mu_y) \wedge \overline{f}(\mu_y, z) \end{aligned}$$

which implies that

$$\begin{aligned} f(x, z) &\geq sup\{\pi(x, \mu_y) \wedge \overline{f}(\mu_y, z) : y \in G\} \\ &= \overline{f} \circ \pi(x, z) \end{aligned}$$

Thus $\overline{f} \circ \pi = f$. Also it is easy to observe that $K_{\overline{f}} = \frac{K_f}{\mu}$. Moreover $\overline{f}$ is a fuzzy isomorphism if and only if $f$ is onto and $G_{(K_f)} = G_\mu$ □

THEOREM 3.15. *Let $f$ be a fuzzy homomorphism from $G$ to $H$ and $\mu$ a normal fuzzy subgroup of $G$, $\nu$ a normal fuzzy subgroup of $H$ such* *that* $f(\mu) \subseteq \nu$. *Then* $f$ *induces a fuzzy homomorphism* $\overline{f}$ *of* $\frac{G}{\mu}$ *into* $\frac{H}{\nu}$ *such that* $\overline{f}$ *is a fuzzy isomorphism if and only if* $R_f \vee \nu = 1_H$ *and* $G_{(f^{-1}(\nu))} \subseteq G_\mu$.

*Proof.* Define a fuzzy relation $\overline{f}$ from $\frac{G}{\mu}$ into $\frac{H}{\nu}$ by:

$$\overline{f}(\mu_x, \nu_y) = f(x, y) \quad \forall x \in G, y \in H$$

then this $\overline{f}$ is a fuzzy homomorphism induced by $f$ such that it is a fuzzy isomorhism if and only if $R_f \vee \nu = 1_H$ and $f^{-1}(\nu) \subseteq \mu$. □

Theorem 3.16. (The First Fuzzy Isomorphism Theorem)
*Let* $f$ *be a fuzzy homomorphism from* $G$ *to* $G'$. *Then*

$$\frac{G}{K_f} \overset{F}{\cong} f(G)$$

*In particular if* $f$ *is a fuzzy epimorphism then*

$$\frac{G}{K_f} \overset{F}{\cong} G'$$

*Proof.* The proof follows from the fundamental theorem of fuzzy homomorphisms by taking $\mu = K_f$. □

Example 3.17. Let $Z$ be the additive group of integers and for $n \geq 1$, let $Z_n$ be the additive group of integers modulo $n$. For each $n$, define a fuzzy subset $\mu_n$ of $Z$ by

$$\mu_n(a) = \begin{cases} 1 & n \textit{ divides } a \\ 0 & \textit{otherwise} \end{cases}$$

for all $a \in Z$. Then $\mu_n$ is a fuzzy normal subgroup of $Z$ and

$$\frac{Z}{\mu_n} \overset{F}{\cong} Z_n$$

*Proof.* Define a fuzzy subset $f$ of $Z \times Z_n$ as follows: for any $a \in Z$, there exists $q \in Z$ and $0 \leq r < n$ such that $a = qn + r$ then for $a \in Z$ and $b \in Z_n$ define:

$$f(a, b) = \begin{cases} 1 & \textit{if b=r} \\ 0 & \textit{otherwise} \end{cases}$$

Then this $f$ is a fuzzy epimorphism of $Z$ onto $Z_n$ such that its fuzzy kernel is precisely $\mu_n$ therefore by the first fuzzy isomorphism theorem we get that

$$\frac{Z}{\mu_n} \overset{F}{\cong} Z_n$$

□

THEOREM 3.18. *If $\mu$ and $\nu$ are fuzzy normal subgroups of $G$ such that $\mu \circ \nu = 1_G$, then*

$$\frac{G}{\mu \cap \nu} \overset{F}{\cong} \frac{G}{\mu} \times \frac{G}{\nu}$$

*Proof.* Define a fuzzy relation $f$ from $G$ into $\frac{G}{\mu} \times \frac{G}{\nu}$ by:

$$f(a, (\mu_x, \nu_y)) = \mu(ax^{-1}) \wedge \nu(ay^{-1})$$

for all $a \in G$ and all $x, y \in G$. Then we get that this $f$ is a fuzzy epimorphism of $G$ onto $\frac{G}{\mu} \times \frac{G}{\nu}$ such that its fuzzy kernel is $\mu \cap \nu$. Therefore by the first fuzzy isomorphism theorem we get

$$\frac{G}{\mu \cap \nu} \overset{F}{\cong} \frac{G}{\mu} \times \frac{G}{\nu}$$

□

THEOREM 3.19. (The Second Fuzzy Isomorphism Theorem)
*Let $\mu$ and $\sigma$ be fuzzy normal subgroups of $G$. Then*

$$\frac{G_\mu}{\mu \cap \sigma} \overset{F}{\cong} \frac{G_\mu G_\sigma}{\sigma}$$

*Proof.* Define a fuzzy relation $f$ from $G_\mu$ into $\frac{G_\mu G_\sigma}{\sigma}$ by:

$$f(x, \sigma_y) = \sigma(xy^{-1}) \ \ \forall\, x \in G_\mu, y \in G_\mu G_\sigma$$

Then $f$ is a fuzzy epimorphism such that its fuzzy kernel is $\mu \cap \sigma$. Thus by the first fuzzy isomorphism theorem we have

$$\frac{G_\mu}{\mu \cap \sigma} \overset{F}{\cong} \frac{G_\mu G_\sigma}{\sigma}$$

□

THEOREM 3.20. *Let $f$ be a fuzzy epimorphism of $G$ onto $H$ and $\nu$ a fuzzy normal subgroup of $H$. Then*

$$\frac{G}{f^{-1}(\nu)} \overset{F}{\cong} \frac{H}{\nu}$$

*Proof.* Define a fuzzy relation $g$ from $G$ into $\frac{H}{\nu}$ as follows:

$$g(x, \nu_y) = sup\{\nu(zy^{-1}) \wedge f(x, z) : z \in H\}$$

for all $x \in G$, $y \in H$. Then it can be worked out that $g$ is a fuzzy epimporphism such that its fuzzy kernel is $f^{-1}(\nu)$. Hence by the first fuzzy isomorphism theorem it follows that

$$\frac{G}{f^{-1}(\nu)} \overset{F}{\cong} \frac{H}{\nu}$$

□

THEOREM 3.21. (The Third Fuzzy Isomorphism Theorem)
*Let $\mu$ and $\sigma$ be fuzzy normal subgroups of $G$ such that $G_\mu \subseteq G_\sigma$. Then*

1. *A fuzzy subset $\frac{\sigma}{\mu}$ of $\frac{G}{\mu}$ defined by:*

   $$\frac{\sigma}{\mu}(\mu_x) = \sigma(x) \text{ for all } x \in G$$

   *is a fuzzy normal subgroup of $\frac{G}{\mu}$.*
2. $$\frac{G/\mu}{\sigma/\mu} \overset{F}{\cong} \frac{G}{\sigma}$$

*Proof.* Define a fuzzy relation $f$ of $\frac{G}{\mu}$ into $\frac{R}{\sigma}$ by:

$$f(\mu_x, \sigma_y) = \sigma(xy^{-1}) \text{ for all } x, y \in G.$$

Then $f$ is a fuzzy epimorphism of $\frac{G}{\mu}$ onto $\frac{G}{\sigma}$. Moreover the fuzzy kernel $K_f$ of $f$ is precisely $\frac{\sigma}{\mu}$. Therefore by the first fuzzy isomorphism theorem we get that

$$\frac{G/\mu}{\sigma/\mu} \overset{F}{\cong} \frac{G}{\sigma}$$

□

THEOREM 3.22. (The correspondence theorem)
*Let $\mu$ be a fuzzy normal subgroup of $G$. Then there is a one-one correspondence between $[\mu, 1_G]$ and $FN(\frac{G}{\mu})$ where:*

$$[\mu, 1_G] = \{\sigma \in FN(G) : G_\mu \subseteq G_\sigma\}$$

*and*

$$FN(\frac{G}{\mu}) = \text{the collection of all fuzzy normal subgroup of } \frac{G}{\mu}$$

*Proof.* It can be shown that the mapping $\sigma \mapsto \frac{\sigma}{\mu}$ of $[\mu, 1_G]$ onto $FN(\frac{G}{\mu})$ is a one to one correspondence. □

Theorem 3.23. *If $f$ is a fuzzy homomorphism of $G$ into an Abelian group $H$ and $\mu$ is a fuzzy subgroup of $G$ containing the fuzzy kernel of $f$, then $\mu$ is fuzzy normal.*

*Proof.* It is enough if we show that $\mu(a^{-1}b^{-1}ab) \geq \mu(a)$ for all $a, b \in G$. For; let $y_a, y_b \in H$ such that $f(a, y_a) = 1 = f(b, y_b)$, so we have $f(a^{-1}b^{-1}ab, y_a^{-1}y_b^{-1}y_a y_b) = 1$. Since $H$ is Abelian it is equivalently saying that $f(a^{-1}b^{-1}ab, e') = 1$. Thus,

$$\begin{aligned} \mu(a^{-1}b^{-1}ab) &\geq K_f(a^{-1}b^{-1}ab) \\ &= f(a^{-1}b^{-1}ab, e') \\ &= 1 \\ &\geq \mu(a) \end{aligned}$$

□

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**Gezahagne Mulat Addis**
Department of Mathematics
University of Gondar
Gondar, Ethiopia
*E-mail*: buttu412@yahoo.com