ON QUASI RICCI SYMMETRIC MANIFOLDS

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ABSTRACT. In this paper, we study a type of Riemannian manifold, namely quasi Ricci symmetric manifold. Among others, we show that the scalar curvature of a quasi Ricci symmetric manifold is constant. In addition if the manifold is Einstein, then its Ricci tensor is zero. Also we prove that if the associated vector field of a quasi Ricci symmetric manifold is either recurrent or concurrent, then its Ricci tensor is zero.

1. Introduction

In [2], Chaki introduced the notion of pseudo Ricci symmetric manifolds such that the Ricci tensor $\text{Ric}$ of a Riemannian manifold $(M^n, g)$ satisfies the relation

$$(\nabla_X \text{Ric})(Y, Z) = 2A(X)\text{Ric}(Y, Z) + A(Y)\text{Ric}(X, Z) + A(Z)\text{Ric}(X, Y)$$

for a nonzero 1-form $A$, where $X, Y, Z \in TM^n$.

A proper example of a pseudo Ricci symmetric manifold is given by Ozen and Altay [4]. On the other hand, in case of conformally flat manifolds, Chaki and Chakrabarti [3] studied several geometric properties of such manifolds. Also in [5], Ray-Guha investigated a conformally flat perfect fluid pseudo Ricci symmetric space time obeying Einstein equation with cosmological constant. Considering this aspect, we study a type
of Riemannian manifold which is called a quasi Ricci symmetric manifold. More precisely, a Riemannian manifold \((M^n, g)\) \((n \geq 3)\) is said to be quasi Ricci symmetric if its Ricci tensor \(\text{Ric}\) fulfills the relation
\[
(\nabla_X \text{Ric})(Y, Z) = 2A(X)\text{Ric}(Y, Z) - A(Y)\text{Ric}(X, Z) - A(Z)\text{Ric}(X, Y),
\]
for a nonzero 1-form \(A\), where \(X, Y, Z \in TM^n\).

The purpose of this paper is to investigate some geometric properties of such a manifold.

2. Main results

The Ricci tensor \(\text{Ric}\) of \((M^n, g)\) is said to be cyclic if it satisfies the relation:
\[
(\nabla_X \text{Ric})(Y, Z) + (\nabla_Y \text{Ric})(Z, X) + (\nabla_Z \text{Ric})(X, Y) = 0.
\]

Now we can state the following:

**Lemma 2.1.** Let \((M^n, g)\) be a quasi Ricci symmetric manifold. Then the Ricci tensor \(\text{Ric}\) of \((M^n, g)\) is cyclic.

**Proof.** By virtue of (1.1) and a straightforward calculation, we can verify that (2.2) holds true.

As a consequence we have

**Theorem 2.2.** Let \((M^n, g)\) be a quasi Ricci symmetric manifold. Then the scalar curvature \(s\) of \((M^n, g)\) is constant.

**Proof.** By Lemma 2.1, we have
\[
(\nabla_X \text{Ric})(Y, Z) + (\nabla_Y \text{Ric})(Z, X) + (\nabla_Z \text{Ric})(X, Y) = 0.
\]

Contracting the last relation on \(Y\) and \(Z\), we obtain
\[
\nabla_X s + 2(\delta \text{Ric})(X) = 0,
\]
which yields from the second Bianchi identity
\[
2\nabla_X s = 0,
\]
showing that the scalar curvature \(s\) of \((M^n, g)\) is constant. This completes the proof.
A vector field $A^\sharp$ on a Riemannian manifold $(M^n, g)$ is called an associated vector field of a 1-form $A$ if $g(X, A^\sharp) = A(X)$ for any $X \in TM^n$.

Concerning the associated vector field $A^\sharp$ of a 1-form $A$ in (1.1), we have

**Lemma 2.3.** Let $(M^n, g)$ be a quasi Ricci symmetric manifold. Then the Ricci tensor $\text{Ric}$ of $(M^n, g)$ satisfies

$$\text{Ric}(X, A^\sharp) = sg(X, A^\sharp).$$

**Proof.** Contracting (1.1) on $Y$ and $Z$, we have

$$\nabla_X s = 2A(X)s - 2\text{Ric}(X, A^\sharp).$$

By virtue of Theorem 2.2, the last relation reduces to

$$0 = 2A(X)s - 2\text{Ric}(X, A^\sharp),$$

which leads to

$$\text{Ric}(X, A^\sharp) = sg(X, A^\sharp).$$

This completes the proof.

As a consequence, we obtain

**Theorem 2.4.** Let $(M^n, g)$ be a quasi Ricci symmetric manifold. If its scalar curvature $s$ of $(M^n, g)$ vanishes, then the Ricci tensor $\text{Ric}$ of $(M^n, g)$ is zero.

**Proof.** Taking account of (1.1) we get

$$\nabla_X(\text{Ric}(Y, Z)) - \text{Ric}(\nabla_X Y, Z) - \text{Ric}(Y, \nabla_X Z)$$

$$= 2A(X)\text{Ric}(Y, Z) - A(Y)\text{Ric}(X, Z) - A(Z)\text{Ric}(X, Y).$$

Putting $Z = A^\sharp$ in the last relation and then using Lemma 2.3, we get

$$\nabla_X (sg(Y, A^\sharp)) - sg(\nabla_X Y, A^\sharp) - sg(Y, \nabla_X A^\sharp)$$

$$= 2A(X)sg(Y, A^\sharp) - A(Y)sg(X, A^\sharp) - g(A^\sharp, A^\sharp)\text{Ric}(X, Y).$$

By virtue of $s = 0$, the last relation reduces to

$$0 = g(A^\sharp, A^\sharp)\text{Ric}(X, Y).$$

Since $g(A^\sharp, A^\sharp) = 0$ is inadmissible by the defining condition of quasi Ricci symmetric manifolds, the last relation implies

$$\text{Ric}(X, Y) = 0.$$

This completes the proof.
A Riemannian manifold \((M^n, g)\) is said to be Einstein if its Ricci tensor \(Ric\) is proportional to the metric tensor \(g\), i.e.,

\[
Ric = \frac{s}{n}g.
\]

Now we can state the following:

**Theorem 2.5.** Let \((M^n, g)\) be a quasi Ricci symmetric manifold. If \((M^n, g)\) is Einstein, then the manifold is Ricci-flat.

**Proof.** By Lemma 2.3, we have

\[
(2.3) \quad Ric(X, A^\sharp) = sg(X, A^\sharp).
\]

On the other hand, by the given Einstein condition, the Ricci tensor \(Ric\) satisfies

\[
(2.4) \quad Ric(X, Y) = \frac{s}{n}g(X, Y).
\]

Putting \(Y = A^\sharp\) in (2.4) and then comparing the relation obtained thus with (2.3), we have

\[
s = 0,
\]

which yields from (2.4)

\[
Ric = 0.
\]

This completes the proof. \(\square\)

The Ricci tensor \(Ric\) of \((M^n, g)\) is said to be of Codazzi type if it satisfies the relation:

\[
(2.5) \quad (\nabla_X Ric)(Y, Z) = (\nabla_Y Ric)(X, Z).
\]

Now we can state the following:

**Theorem 2.6.** Let \((M^n, g)\) be a quasi Ricci symmetric manifold. If its Ricci tensor \(Ric\) is of Codazzi type, then the Ricci tensor \(Ric\) satisfies

\[
Ric(X, Y) = sU(X)U(Y),
\]

where \(U = \frac{A}{\|A\|} \).
Proof. Taking account of (1.1) and (2.5), we have

(2.6) \[ A(X)Ric(Y,Z) = A(Y)Ric(X,Z), \]

which implies

\[ Ric(X,Y) = fA(X)A(Y). \]

Therefore from the last relation it follows that

\[ Ric(X,Y) = sU(X)U(Y), \]

where \( U = \frac{A}{||A||} \). This completes the proof. \( \square \)

A Riemannian manifold \((M^n, g)(n > 3)\) is said to be conformally flat \([1]\) if its curvature tensor \(R\) satisfies the relation:

\[
R(X,Y,Z,W) = \frac{1}{n-2}(Ric(Y,Z)g(X,W) - Ric(Y,W)g(X,Z) + g(Y,Z)Ric(X,W) - g(Y,W)Ric(X,Z)) - g(Y,W)Ric(X,Z)) - \frac{s}{(n-1)(n-2)}(g(Y,Z)g(X,W) - g(Y,W)g(X,Z)).
\]

It is well known \([1]\) that a conformally flat manifold satisfies the relation:

(2.7) \[
(\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) = \frac{1}{2(n-1)}[g(Y,Z)ds(X) - g(X,Z)ds(Y)].
\]

Now we can state the following:

**Theorem 2.7.** Let \((M^n, g)(n > 3)\) be a quasi Ricci symmetric manifold. If the manifold is conformally flat, then the Ricci tensor \(Ric\) of \((M^n, g)\) satisfies

\[ Ric(X,Y) = sU(X)U(Y), \]

where \( U = \frac{A}{||A||} \).

Proof. By virtue of (2.7) and Theorem 2.2, we have

\[
(\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) = 0,
\]

showing that the Ricci tensor of \((M^n, g)\) is of Codazzi type. Therefore it follows from Theorem 2.6 that its Ricci tensor \(Ric\) satisfies

\[ Ric(X,Y) = sU(X)U(Y), \]

where \( U = \frac{A}{||A||} \). This completes the proof. \( \square \)
A vector field $V$ on a Riemannian manifold $(M^n, g)$ is said to be recurrent if it satisfies the relation

$$(\nabla_X V) = \omega(X)V,$$

where $\omega$ is a closed 1-form, i.e., $d\omega = 0$.

Concerning a recurrent vector field $A^\sharp$, we get

**Theorem 2.8.** Let $(M^n, g)$ be a quasi Ricci symmetric manifold. If the associated vector field $A^\sharp$ of a 1-form $A$ in (1.1) is recurrent, then the Ricci tensor $Ric$ of $(M^n, g)$ vanishes.

**Proof.** From the definition of recurrent vector field $A^\sharp$, it follows that

$$R(X,Y)A^\sharp = \nabla_X \nabla_Y A^\sharp - \nabla_Y \nabla_X A^\sharp - \nabla_{[X,Y]} A^\sharp$$

$$= d\omega(X,Y)A^\sharp + \omega(Y)\omega(X)A^\sharp - \omega(X)\omega(Y)A^\sharp = 0.$$

Therefore we obtain

$$g(R(X,Y)A^\sharp, Z) = R(X,Y, A^\sharp, Z) = 0,$$

which yields from contracting on $X$ and $Z$

$$Ric(Y, A^\sharp) = 0.$$  

By virtue of Lemma 2.3 and last identity, we get

$$s = 0,$$

which yields from Theorem 2.4

$$Ric = 0.$$  

This completes the proof. 

A vector field $V$ on a Riemannian manifold $(M^n, g)$ is said to be concurrent if it satisfies the relation

$$(\nabla_X V) = kX,$$

where $k$ is constant.

Concerning a concurrent vector field $A^\sharp$, we have

**Theorem 2.9.** Let $(M^n, g)$ be a quasi Ricci symmetric manifold. If the associated vector field $A^\sharp$ of a 1-form $A$ in (1.1) is concurrent, then the Ricci tensor $Ric$ of $(M^n, g)$ vanishes.
Proof. From the definition of concurrent vector field $A^\sharp$, it follows that
\[
R(X,Y)A^\sharp = \nabla_X \nabla_Y A^\sharp - \nabla_Y \nabla_X A^\sharp - \nabla_{[X,Y]} A^\sharp = k(\nabla_X Y - \nabla_Y X - [X,Y]) = 0.
\]
Therefore we obtain
\[
g(R(X,Y)A^\sharp, Z) = R(X,Y, A^\sharp, Z) = 0,
\]
which yields from contracting on $X$ and $Z$
\[
\text{Ric}(Y, A^\sharp) = 0.
\]
By virtue of Lemma 2.3 and last identity, we get
\[
s = 0,
\]
which yields from Theorem 2.4
\[
\text{Ric} = 0.
\]
This completes the proof. \qed

References


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