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ON QUASI RICCI SYMMETRIC MANIFOLDS

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Abstract. In this paper, we study a type of Riemannian manifold, namely quasi Ricci symmetric manifold. Among others, we show that the scalar curvature of a quasi Ricci symmetric manifold is constant. In addition if the manifold is Einstein, then its Ricci tensor is zero. Also we prove that if the associated vector field of a quasi Ricci symmetric manifold is either recurrent or concurrent, then its Ricci tensor is zero.

1. Introduction

In [2], Chaki introduced the notion of pseudo Ricci symmetric manifolds such that the Ricci tensor Ric of a Riemannian manifold (M^n, g) satisfies the relation

$$(\nabla_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) + A(Y)Ric(X,Z) + A(Z)Ric(X,Y)$$

for a nonzero 1-form A, where $X, Y, Z \in TM^n$.
A proper example of a pseudo Ricci symmetric manifold is given by Ozen and Altay [4]. On the other hand, in case of conformally flat manifolds, Chaki and Chakrabarti [3] studied several geometric properties of such manifolds. Also in [5], Ray-Guha investigated a conformally flat perfect fluid pseudo Ricci symmetric space time obeying Einstein equation with cosmological constant. Considering this aspect, we study a type

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of Riemannian manifold which is called a quasi Ricci symmetric manifold. More precisely, a Riemannian manifold $(M^n, g)(n \geq 3)$ is said to be quasi Ricci symmetric if its Ricci tensor Ric fulfills the relation

$$(\nabla_X Ric)(Y,Z) = 2A(X)Ric(Y,Z) - A(Y)Ric(X,Z) - A(Z)Ric(X,Y), \tag{1.1}$$

for a nonzero 1-form A, where $X, Y, Z \in TM^n$.
The purpose of this paper is to investigate some geometric properties of such a manifold.

2. Main results

The Ricci tensor Ric of (M^n, g) is said to be cyclic if it satisfies the relation:

$$(\nabla_X Ric)(Y,Z) + (\nabla_Y Ric)(Z,X) + (\nabla_Z Ric)(X,Y) = 0. \tag{2.2}$$

Now we can state the following:

LEMMA 2.1. *Let (M^n, g) be a quasi Ricci symmetric manifold. Then the Ricci tensor Ric of (M^n, g) is cyclic.*

Proof. By virtue of (1.1) and a straightforward calculation, we can verify that (2.2) holds true. □

As a consequence we have

THEOREM 2.2. *Let (M^n, g) be a quasi Ricci symmetric manifold. Then the scalar curvature s of (M^n, g) is constant.*

Proof. By Lemma 2.1, we have

$$(\nabla_X Ric)(Y,Z) + (\nabla_Y Ric)(Z,X) + (\nabla_Z Ric)(X,Y) = 0.$$

Contracting the last relation on Y and Z, we obtain

$$\nabla_X s + 2(\delta Ric)(X) = 0,$$

which yields from the second Bianchi identity

$$2\nabla_X s = 0,$$

showing that the scalar curvature s of (M^n, g) is constant. This completes the proof. □

A vector field $A^\sharp$ on a Riemannian manifold (M^n, g) is called an associated vector field of a 1-form A if $g(X, A^\sharp) = A(X)$ for any $X \in TM^n$. Concerning the associated vector field $A^\sharp$ of a 1-form A in (1.1), we have

LEMMA 2.3. *Let (M^n, g) be a quasi Ricci symmetric manifold. Then the Ricci tensor Ric of (M^n, g) satisfies*

$$Ric(X, A^\sharp) = sg(X, A^\sharp).$$

Proof. Contracting (1.1) on Y and Z, we have

$$\nabla_X s = 2A(X)s - 2Ric(X, A^\sharp).$$

By virtue of Theorem 2.2, the last relation reduces to

$$0 = 2A(X)s - 2Ric(X, A^\sharp),$$

which leads to

$$Ric(X, A^\sharp) = sg(X, A^\sharp).$$

This completes the proof. □

As a consequence, we obtain

THEOREM 2.4. *Let (M^n, g) be a quasi Ricci symmetric manifold. If its scalar curvature s of (M^n, g) vanishes, then the Ricci tensor Ric of (M^n, g) is zero.*

Proof. Taking account of (1.1) we get

$$\nabla_X(Ric(Y, Z)) - Ric(\nabla_X Y, Z) - Ric(Y, \nabla_X Z)$$
$$= 2A(X)Ric(Y, Z) - A(Y)Ric(X, Z) - A(Z)Ric(X, Y).$$

Putting $Z = A^\sharp$ in the last relation and then using Lemma 2.3, we get

$$\nabla_X(sg(Y, A^\sharp)) - sg(\nabla_X Y, A^\sharp) - sg(Y, \nabla_X A^\sharp)$$
$$= 2A(X)sg(Y, A^\sharp) - A(Y)sg(X, A^\sharp) - g(A^\sharp, A^\sharp)Ric(X, Y).$$

By virtue of $s = 0$, the last relation reduces to

$$0 = g(A^\sharp, A^\sharp)Ric(X, Y).$$

Since $g(A^\sharp, A^\sharp) = 0$ is inadmissible by the defining condition of quasi Ricci symmetric manifolds, the last relation implies

$$Ric(X, Y) = 0.$$

This completes the proof. □

A Riemannian manifold (M^n, g) is said to be Einstein if its Ricci tensor *Ric* is proportional to the metric tensor g, i.e.,

$$Ric = \frac{s}{n} g.$$

Now we can state the following:

THEOREM 2.5. *Let (M^n, g) be a quasi Ricci symmetric manifold. If (M^n, g) is Einstein, then the manifold is Ricci-flat.*

Proof. By Lemma 2.3, we have

$$Ric(X, A^\sharp) = sg(X, A^\sharp). \tag{2.3}$$

On the other hand, by the given Einstein condition, the Ricci tensor *Ric* satisfies

$$Ric(X, Y) = \frac{s}{n} g(X, Y). \tag{2.4}$$

Putting $Y = A^\sharp$ in (2.4) and then comparing the relation obtained thus with (2.3), we have

$$s = 0,$$

which yields from (2.4)

$$Ric = 0.$$

This completes the proof. □

The Ricci tensor *Ric* of (M^n, g) is said to be of Codazzi type if it satisfies the relation:

$$(\nabla_X Ric)(Y, Z) = (\nabla_Y Ric)(X, Z). \tag{2.5}$$

Now we can state the following:

THEOREM 2.6. *Let (M^n, g) be a quasi Ricci symmetric manifold. If its Ricci tensor Ric is of Codazzi type, then the Ricci tensor Ric satisfies*

$$Ric(X, Y) = sU(X)U(Y),$$

where $U = \frac{A}{||A||}$.

Proof. Taking account of (1.1) and (2.5), we have

$$A(X)Ric(Y,Z) = A(Y)Ric(X,Z), \tag{2.6}$$

which implies

$$Ric(X,Y) = fA(X)A(Y).$$

Therefore from the last relation it follows that

$$Ric(X,Y) = sU(X)U(Y),$$

where $U = \frac{A}{||A||}$. This completes the proof. □

A Riemannian manifold $(M^n, g)(n > 3)$ is said to be conformally flat [1] if its curvature tensor R satisfies the relation:

$$R(X,Y,Z,W) = \frac{1}{n-2}(Ric(Y,Z)g(X,W) - Ric(Y,W)g(X,Z) + g(Y,Z)Ric(X,W)$$

$$-g(Y,W)Ric(X,Z)) - \frac{s}{(n-1)(n-2)}(g(Y,Z)g(X,W) - g(Y,W)g(X,Z)).$$

It is well known [1] that a conformally flat manifold satisfies the relation:

(2.7)

$$(\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) = \frac{1}{2(n-1)}[g(Y,Z)ds(X) - g(X,Z)ds(Y)].$$

Now we can state the following:

Theorem 2.7. *Let $(M^n, g)(n > 3)$ be a quasi Ricci symmetric manifold. If the manifold is conformally flat, then the Ricci tensor Ric of (M^n, g) satisfies*

$$Ric(X,Y) = sU(X)U(Y),$$

where $U = \frac{A}{||A||}$.

Proof. By virtue of (2.7) and Theorem 2.2, we have

$$(\nabla_X Ric)(Y,Z) - (\nabla_Y Ric)(X,Z) = 0,$$

showing that the Ricci tensor of (M^n, g) is of Codazzi type. Therefore it follows from Theorem 2.6 that its Ricci tensor *Ric* satisfies

$$Ric(X,Y) = sU(X)U(Y),$$

where $U = \frac{A}{||A||}$. This completes the proof. □

A vector field V on a Riemannian manifold (M^n, g) is said to be recurrent if it satisfies the relation

$$(\nabla_X V) = \omega(X)V,$$

where ω is a closed 1-form, i.e., $d\omega = 0$.
Concerning a recurrent vector field $A^\sharp$, we get

THEOREM 2.8. *Let (M^n, g) be a quasi Ricci symmetric manifold. If the associated vector field $A^\sharp$ of a 1-form A in (1.1) is recurrent, then the Ricci tensor Ric of (M^n, g) vanishes.*

Proof. From the definition of recurrent vector field $A^\sharp$, it follows that

$$R(X,Y)A^\sharp = \nabla_X \nabla_Y A^\sharp - \nabla_Y \nabla_X A^\sharp - \nabla_{[X,Y]} A^\sharp$$
$$= d\omega(X,Y)A^\sharp + \omega(Y)\omega(X)A^\sharp - \omega(X)\omega(Y)A^\sharp = 0.$$

Therefore we obtain

$$g(R(X,Y)A^\sharp, Z) = R(X,Y,A^\sharp,Z) = 0,$$

which yields from contracting on X and Z

$$Ric(Y, A^\sharp) = 0.$$

By virtue of Lemma 2.3 and last identity, we get

$$s = 0,$$

which yields from Theorem 2.4

$$Ric = 0.$$

This completes the proof. □

A vector field V on a Riemannian manifold (M^n, g) is said to be concurrent if it satisfies the relation

$$(\nabla_X V) = kX,$$

where k is constant.
Concerning a concurrent vector field $A^\sharp$, we have

THEOREM 2.9. *Let (M^n, g) be a quasi Ricci symmetric manifold. If the associated vector field $A^\sharp$ of a 1-form A in (1.1) is concurrent, then the Ricci tensor Ric of (M^n, g) vanishes.*

Proof. From the definition of concurrent vector field $A^\sharp$, it follows that

$$R(X,Y)A^\sharp = \nabla_X \nabla_Y A^\sharp - \nabla_Y \nabla_X A^\sharp - \nabla_{[X,Y]} A^\sharp$$
$$= k(\nabla_X Y - \nabla_Y X - [X,Y]) = 0.$$

Therefore we obtain

$$g(R(X,Y)A^\sharp, Z) = R(X,Y,A^\sharp,Z) = 0,$$

which yields from contracting on X and Z

$$Ric(Y, A^\sharp) = 0.$$

By virtue of Lemma 2.3 and last identity, we get

$$s = 0,$$

which yields from Theorem 2.4

$$Ric = 0.$$

This completes the proof. □

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