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THE GROWTH OF ENTIRE FUNCTION IN THE FORM OF VECTOR VALUED DIRICHLET SERIES IN TERMS OF (p, q)-TH RELATIVE RITT ORDER AND (p, q)-TH RELATIVE RITT TYPE

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Abstract. In this paper we wish to study some growth properties of entire functions represented by a vector valued Dirichlet series on the basis of (p, q)-th relative Ritt order, (p, q)-th relative Ritt type and (p, q)-th relative Ritt weak type where p and q are integers such that $p \geq 0$ and $q \geq 0$.

1. Introduction and preliminaries

Suppose $f(s)$ be an entire function of the complex variable $s = \sigma + it$ (σ and t are real variables) defined by everywhere absolutely convergent vector valued Dirichlet series briefly known as VVDS

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n} \tag{1}$$

where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n's are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}\,(n \geq 1)\,, \lambda_n \to +\infty$ as $n \to +\infty$ and satisfy the conditions $\overline{\lim\limits_{n\to+\infty}} \frac{\log n}{\lambda_n} = D < +\infty$ and $\overline{\lim\limits_{n\to+\infty}} \frac{\log\|a_n\|}{\lambda_n} =$

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$-\infty$. If σ_c and σ_a denote respectively the abscissa of convergence and absolute convergence of (1), then in this case clearly $\sigma_a = \sigma_c = +\infty$. The function $M_f(\sigma)$ known as maximum modulus function corresponding to an entire function $f(s)$ defined by (1), is written as follows

$$M_f(\sigma) = \underset{-\infty<t<+\infty}{l.u.b.} \|f(\sigma + it)\| .$$

Now we state the following two notations which are frequently use in our subsequent study:

$$\begin{aligned} \log^{[k]} x &= \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ; \\ \log^{[0]} x &= x, \log^{[-1]} x = \exp x \end{aligned}$$

and

$$\begin{aligned} \exp^{[k]} x &= \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \cdots ; \\ \exp^{[0]} x &= x, \exp^{[-1]} x = \log x. \end{aligned}$$

Juneja, Nandan and Kapoor [10] first introduced the concept of (p, q)-th order and (p, q)-th lower order of an entire Dirichlet series where $p \geq q + 1 \geq 1$. In the line of Juneja et al. [10], one can define the (p, q) -th Ritt order (respectively (p, q)-th Ritt lower order) of an entire function f represented by VVDS in the following way:

$$\rho^{(p,q)}(f) = \varlimsup_{\sigma\to+\infty} \frac{\log^{[p]} M_f(\sigma)}{\log^{[q]} \sigma} = \varlimsup_{\sigma\to+\infty} \frac{\log^{[p]} \sigma}{\log^{[q]} M_f^{-1}(\sigma)}$$

$$\left(\text{respectively } \lambda^{(p,q)}(f) = \varliminf_{\sigma\to+\infty} \frac{\log^{[p]} M_f(\sigma)}{\log^{[q]} \sigma} = \varliminf_{\sigma\to+\infty} \frac{\log^{[p]} \sigma}{\log^{[q]} M_f^{-1}(\sigma)}\right),$$

where p and q are integers such that $p \geq q + 1 \geq 1$.

In this connection let us recall that if $0 < \rho^{(p,q)}(f) < \infty$, then the following properties hold

$$\begin{cases} \rho^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases} .$$

Similarly for $0 < \lambda^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases} .$$

An entire function f (represented by VVDS) of index-pair (p,q) is said to be of regular (p,q) Ritt growth if its (p,q)-th Ritt order coincides with its (p,q)-th Ritt lower order, otherwise f is said to be of irregular (p,q) Ritt growth.

Now to compare the relative growth of two entire functions represented by VVDS having same nonzero finite (p,q)-th Ritt order, one may introduce the definition of (p,q)-th Ritt type (respectively (p,q)-th Ritt lower type) in the following manner:

Definition 1. The (p,q)-th Ritt type and (p,q)-th Ritt lower type respectively denoted by $\Delta^{(p,q)}(f)$ and $\overline{\Delta}^{(p,q)}(f)$ of an entire function f represented by VVDS when $0<\rho_f(p,q)<+\infty$ are defined as follows:

$$\Delta^{(p,q)}(f)=\overline{\lim_{\sigma\to+\infty}}\frac{\log^{[p-1]}M_f(\sigma)}{\left[\log^{[q-1]}\sigma\right]^{\rho^{(p,q)}(f)}}\text{ and }\overline{\Delta}^{(p,q)}(f)=\underline{\lim_{\sigma\to+\infty}}\frac{\log^{[p-1]}M_f(\sigma)}{\left[\log^{[q-1]}\sigma\right]^{\rho^{(p,q)}(f)}}$$

where p and q are integers such that $p\geq q+1\geq 1$.

Analogously to determine the relative growth of two entire functions represented by vector valued Dirichlet series having same nonzero finite (p,q)-th Ritt lower order, one may introduce the definition of (p,q)-th Ritt weak type in the following way:

Definition 2. The (p,q)- th Ritt weak type denoted by $\tau^{(p,q)}(f)$ of an entire function f represented by VVDS is defined as follows:

$$\tau^{(p,q)}(f)=\underline{\lim_{\sigma\to+\infty}}\frac{\log^{[p-1]}M_f(\sigma)}{\left[\log^{[q-1]}\sigma\right]^{\lambda^{(p,q)}(f)}},\ 0<\ \lambda_f(p,q)\ <+\infty$$

where p and q are integers such that $p\geq q+1\geq 1$.

Also one may define the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an entire function f represented by VVDS in the following manner :

$$\overline{\tau}^{(p,q)}(f)=\overline{\lim_{\sigma\to+\infty}}\frac{\log^{[p-1]}M_f(\sigma)}{\left[\log^{[q-1]}\sigma\right]^{\lambda^{(p,q)}(f)}},\ 0<\ \lambda_f(p,q)\ <+\infty,$$

where p and q are integers such that $p\geq q+1\geq 1$.

G. S. Srivastava [14] introduced the relative Ritt order between two entire functions represented by VVDS to avoid comparing growth just with $\exp\exp s$ In the case of relative Ritt order, it therefore seems reasonable to define suitably the (p,q)-th relative Ritt order of entire function represented by VVDS. Recently, Datta and Biswas [6] introduce the concept of (p,q)-th relative Ritt order$\rho_g^{(p,q)}(f)$ of an entire function f represented by VVDS with respect to another entire function g which is also represented by VVDS, in the light of index-pair which is as follows:

DEFINITION 3. [6] Let f and g be any two entire functions represented by VVDS with index-pair (m,q) and (m,p), respectively, where p,q,m are integers such that $m \geq q+1 \geq 1$ and $m \geq p+1 \geq 1$. Then the (p,q)-th relative Ritt order and (p,q)-th relative Ritt lower order of f with respect to g are defined as

$$\rho_g^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(\sigma))}{\log^{[q]}\sigma} = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_g^{-1}(\sigma)}{\log^{[q]} M_f^{-1}(\sigma)}$$

and

$$\lambda_g^{(p,q)}(f) = \underline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(\sigma))}{\log^{[q]}\sigma} = \underline{\lim_{\sigma\to+\infty}} \frac{\log^{[p]} M_g^{-1}(\sigma)}{\log^{[q]} M_f^{-1}(\sigma)}.$$

In this connection, the following definition is relevant:

DEFINITION 4. [3] Let f and g be any two entire functions both represented by VVDS with index-pairs (m,q) and (m,p) respectively where p,q,m are integers such that $m \geq q+1 \geq 1$ and $m \geq p+1 \geq 1$. Then the entire function f is said to have relative index-pair (p,q) with respect to another entire function g, if $b < \rho_g^{(p,q)}(f) < \infty$ and $\rho_g^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b=1$ if $p=q=m$ and $b=0$ otherwise. Moreover if $0 < \rho_g^{(p,q)}(f) < \infty$, then

$$\begin{cases} \rho_g^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \rho_g^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho_g^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1,2,\cdots . \end{cases}$$

Similarly for $0 < \lambda_g^{(p,q)}(f) < \infty$, one can easily verify that

$$\begin{cases} \lambda_g^{(p-n,q)}(f) = \infty & \text{for} \quad n < p, \\ \lambda_g^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda_g^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1,2,\cdots . \end{cases}$$

Further an entire function f (represented by VVDS) for which (p,q)-th relative Ritt order and (p,q)-th relative Ritt lower order with respect to another entire function g (represented by VVDS) are the same is called a function of regular relative (p,q) Ritt growth with respect to g. Otherwise, f is said to be irregular relative (p,q) Ritt growth.with respect to g.

Now in order to compare the relative growth of two entire functions represented by VVDS having same nonzero finite (p,q)-th relative Ritt order with respect to another entire function represented by VVDS, one may introduce the concepts of (p,q)-th relative Ritt-type (respectively (p,q)-th relative Ritt lower type) which are as follows:

Definition 5. [2] Let f and g be any two entire functions represented by VVDS with index-pair (m,q) and (m,p), respectively, where p,q,m are integers such that $m \geq q+1 \geq 1$ and $m \geq p+1 \geq 1$ and $0 < \rho_g^{(p,q)}(f) < +\infty$. Then the (p,q)-th relative Ritt type and (p,q)-th relative Ritt lower type of f with respect to g are defined as

$$\Delta_g^{(p,q)}(f) = \overline{\lim_{\sigma \to +\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]} \sigma\right]^{\rho_g^{(p,q)}(f)}} \text{ and}$$

$$\overline{\Delta}_g^{(p,q)}(f) = \underline{\lim}_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]} \sigma\right]^{\rho_g^{(p,q)}(f)}}.$$

Analogously to determine the relative growth of two entire functions represented by VVDS having same nonzero finite $p,q)$-th relative Ritt lower order with respect to another entire function represented by VVDS, one may introduce the definition of (p,q)-th relative Ritt weak type in the following way:

Definition 6. [2] Let f and g be any two entire functions represented by VVDS with index-pair (m,q) and (m,p), respectively, where p,q,m are integers such that $m \geq q+1 \geq 1$ and $m \geq p+1 \geq 1$. Then (p,q)-th relative Ritt weak type denoted by $\tau_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g is defined as follows:

$$\tau_g^{(p,q)}(f) = \underline{\lim}_{\sigma \to +\infty} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]} \sigma\right]^{\lambda_g^{(p,q)}(f)}}, \; 0 < \lambda_g^{(p,q)}(f) < +\infty.$$

Similarly the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g both represented by VVDS in the following manner :

$$\overline{\tau}_g^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\lambda_g^{(p,q)}(f)}},\ 0 < \ \lambda_g^{(p,q)}(f)\ \ < +\infty.$$

During the past decades, several authors (see [1–5,11–13,15–17]) made closed investigations on the properties of entire Dirichlet series in different directions using the growth indicator such as Ritt order. In the present paper we wish to establish some basic properties of entire functions represented by a VVDS on the basis of (p,q)-th relative Ritt order, (p,q)-th relative Ritt type and (p,q)-th relative Ritt weak type where p and q are integers such that $p \geq 0$ and $q \geq 0$. Through out the paper we consider that all the growth indicators are nonzero finite.

2. Main Results

In this section we state the main results of the paper.

Theorem 1. *Let f, g and h be any three entire functions represented by vector valued Dirichlet series. Also let p,q,m are integers such that $p \geq 0$, $q \geq 0$ and $m \geq 0$. If (m,q)-th relative Ritt order (respectively (m,q)-th relative Ritt lower order) of f with respect to h and (m,p)-th relative Ritt order (respectively (m,p)-th relative Ritt lower order) of g with respect to h are respectively denoted by $\rho_h^{(m,q)}(f)$ $\left(\text{respectively } \lambda_h^{(m,q)}(f)\right)$ and $\rho_h^{(m,p)}(g)$ $\left(\text{respectively } \lambda_h^{(m,p)}(g)\right)$, then*

$$\frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \lambda_g^{(p,q)}(f) \leq \min\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\right\}$$
$$\leq \max\left\{\frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}, \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}\right\} \leq \rho_g^{(p,q)}(f) \leq \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}\ .$$

Proof. From the definitions of $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ we get that

$$\text{(2)}\qquad \log\rho_g^{(p,q)}(f) = \overline{\lim_{\sigma\to+\infty}}\left(\log^{[p+1]} M_g^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma)\right),$$

$$(3)\qquad \log \lambda_g^{(p,q)}(f) = \varliminf_{\sigma\to+\infty}\left(\log^{[p+1]} M_g^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma)\right).$$

Now from the definitions of $\rho_h^{(m,q)}(f)$ and $\lambda_h^{(m,q)}(f)$, it follows that

$$(4)\quad \log \rho_h^{(m,q)}(f) = \varlimsup_{\sigma\to+\infty}\left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma)\right),$$

$$(5)\quad \log \lambda_h^{(m,q)}(f) = \varliminf_{\sigma\to+\infty}\left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma)\right).$$

Similarly, from the definitions of $\rho_h^{(m,p)}(g)$ and $\lambda_h^{(m,p)}(g)$, we obtain that

$$(6)\quad \log \rho_h^{(m,p)}(g) = \varlimsup_{\sigma\to+\infty}\left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[p+1]} M_g^{-1}(\sigma)\right),$$

$$(7)\quad \log \lambda_h^{(m,p)}(g) = \varliminf_{\sigma\to+\infty}\left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[p+1]} M_g^{-1}(\sigma)\right).$$

Therefore from (3), (5) and (6), we get that

$$\log \lambda_g^{(p,q)}(f) = \varliminf_{\sigma\to+\infty}\Big[\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma) - \left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[p+1]} M_g^{-1}(\sigma)\right)\Big]$$

$$i.e.,\ \log \lambda_g^{(p,q)}(f) \geq \Big[\varliminf_{\sigma\to+\infty}\left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma)\right) - \varlimsup_{\sigma\to+\infty}\left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[p+1]} M_g^{-1}(\sigma)\right)\Big]$$

$$(8)\qquad i.e.,\ \log \lambda_g^{(p,q)}(f) \geq \left(\log \lambda_h^{(m,q)}(f) - \log \rho_h^{(m,p)}(g)\right).$$

Similarly, from (2), (4) and (7), it follows that

$$\log \rho_g^{(p,q)}(f) = \varlimsup_{\sigma\to+\infty}\Big[\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma) - \left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[p+1]} M_g^{-1}(\sigma)\right)\Big]$$

$$i.e.,\ \log \rho_g^{(p,q)}(f) \leq \Big[\varlimsup_{\sigma\to+\infty}\left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma)\right) - \varliminf_{\sigma\to+\infty}\left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[p+1]} M_g^{-1}(\sigma)\right)\Big]$$

(9) $$i.e.,\ \log \rho_g^{(p,q)}(f) \le \left(\log \rho_h^{(m,q)}(f) - \log \lambda_h^{(m,p)}(g)\right).$$

Again, in view of (3) we obtain that

$$\log \lambda_g^{(p,q)}(f) = \varliminf_{\sigma\to\infty}\Big[\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma) - \left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[p+1]} M_g^{-1}(\sigma)\right)\Big]$$

By taking $A = \left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma)\right)$ and $B = \left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[p+1]} M_g^{-1}(\sigma)\right)$, we get from above that

$$\log \lambda_g^{(p,q)}(f) \le \min\left(\varliminf_{\sigma\to+\infty} A + \varlimsup_{\sigma\to+\infty} -B, \varlimsup_{\sigma\to+\infty} A + \varliminf_{\sigma\to+\infty} -B\right)$$

$$i.e.,\ \log \lambda_g^{(p,q)}(f) \le \min\left(\varliminf_{\sigma\to\infty} A - \varliminf_{\sigma\to\infty} B, \varlimsup_{\sigma\to\infty} A - \varlimsup_{\sigma\to\infty} B\right).$$

Therefore in view of (4), (5), (6) and (7) we get from above that

(10) $$\log \lambda_g^{(p,q)}(f) \le \min\left(\log \lambda_h^{(m,q)}(f) - \log \lambda_h^{(m,p)}(g), \log \rho_h^{(m,q)}(f) - \log \rho_h^{(m,p)}(g)\right).$$

Further from (2) it follows that

$$\log \rho_g^{(p,q)}(f) = \varlimsup_{\sigma\to+\infty}\Big[\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma) - \left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[p+1]} M_g^{-1}(\sigma)\right)\Big]$$

By taking $A = \left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[q+1]} M_f^{-1}(\sigma)\right)$ and $B = \left(\log^{[m+1]} M_h^{-1}(\sigma) - \log^{[p+1]} M_g^{-1}(\sigma)\right)$, we obtain from above that

$$\log \rho_g^{(p,q)}(f) \ge \max\left(\varliminf_{\sigma\to+\infty} A + \varlimsup_{\sigma\to+\infty} -B, \varlimsup_{\sigma\to+\infty} A + \varliminf_{\sigma\to+\infty} -B\right)$$

$$i.e.,\ \log \rho_g^{(p,q)}(f) \ge \max\left(\varliminf_{\sigma\to+\infty} A - \varliminf_{\sigma\to+\infty} B, \varlimsup_{\sigma\to+\infty} A - \varlimsup_{\sigma\to+\infty} B\right).$$

Therefore in view of (4), (5), (6) and (7), it follows from above that

(11) $$\log \rho_g^{(p,q)}(f) \ge \max\left(\log \lambda_h^{(m,q)}(f) - \log \lambda_h^{(m,p)}(g), \log \rho_h^{(m,q)}(f) - \log \rho_h^{(m,p)}(g)\right).$$

Thus the theorem follows from (8) , (9) , (10) and (11) . □

In view of Theorem 1, one can easily verify the following corollaries:

Corollary 1. *Let f, g and h be any three entire functions represented by vector valued Dirichlet series. Also let f be an entire function with regular relative (m, q) Ritt growth with respect to entire function h and g be entire having relative index-pair (m, p) with respect to another entire function h where p, q, m are integers such that $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then*

$$\lambda_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \quad \text{and} \quad \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}.$$

In addition, if $\rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g)$, then

$$\lambda_g^{(p,q)}(f) = \rho_f^{(q,p)}(g) = 1.$$

Corollary 2. *Let f, g and h be any three entire functions represented by vector valued Dirichlet series. Also let f be an entire function with relative index-pair (m, q) with respect to entire function h and g be entire of regular relative (m, p) Ritt growth with respect to another entire function h where p, q, m are integers such that $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then*

$$\lambda_g^{(p,q)}(f) = \frac{\lambda_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \quad \text{and} \quad \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}.$$

In addition, if $\rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g)$, then

$$\rho_g^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = 1.$$

Corollary 3. *Let f, g and h be any three entire functions represented by vector valued Dirichlet series. Also let f and g be any two entire functions with regular relative (m, q) Ritt growth and regular relative (m, p) Ritt growth with respect to entire function h respectively where p, q, m are integers such that $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then*

$$\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}.$$

Corollary 4. *Let f, g and h be any three entire functions represented by vector valued Dirichlet series. Also let f and g be any two entire functions with rregular relative (m, q) Ritt growth and regular* *relative* (m,p) *Ritt growth with respect to entire function* h *respectively where* p,q,m *are integers such that* $p \geq 0$, $q \geq 0$ *and* $m \geq 0$. *Also suppose that* $\rho_h^{(m,q)}(f) = \rho_h^{(m,p)}(g)$. *Then*

$$\lambda_g^{(p,q)}(f) = \rho_g^{(p,q)}(f) = \lambda_f^{(q,p)}(g) = \rho_f^{(q,p)}(g) = 1.$$

COROLLARY 5. *Let* f, g *and* h *be any three entire functions represented by vector valued Dirichlet series. Also let* f *and* g *be any two entire functions with relative index-pairs* (m,q) *and* (m,p) *with respect to entire function* h *respectively where* p,q,m *are integers such that* $p \geq 0$, $q \geq 0$, $m \geq 0$ *and either* f *is not of regular relative* (m,q) *Ritt growth or* g *is not of regular relative* (m,p) *Ritt growth, then*

$$\rho_g^{(p,q)}(f) . \rho_f^{(q,p)}(g) \geq 1.$$

If f *and* g *are both of regular relative* (m,q) *Ritt growth and regular relative* (m,p) *Ritt growth with respect to entire function* h *respectively, then*

$$\rho_g^{(p,q)}(f) . \rho_f^{(q,p)}(g) = 1.$$

COROLLARY 6. *Let* f, g *and* h *be any three entire functions represented by vector valued Dirichlet series. Also let* f *and* g *be any two entire functions with relative index-pairs* (m,q) *and* (m,p) *with respect to entire function* h *respectively where* p,q,m *are integers such that* $p \geq 0$, $q \geq 0$, $m \geq 0$ *and either* f *is not of regular relative* (m,q) *Ritt growth or* g *is not of regular relative* (m,p) *Ritt growth, then*

$$\lambda_g^{(p,q)}(f) . \lambda_f^{(q,p)}(g) \leq 1.$$

If f *and* g *are both of regular relative* (m,q) *Ritt growth and regular relative* (m,p) *Ritt growth with respect to entire function* h *respectively, then*

$$\lambda_g^{(p,q)}(f) . \lambda_f^{(q,p)}(g) = 1.$$

COROLLARY 7. *Let* f, g *and* h *be any three entire functions represented by vector valued Dirichlet series. Also let* p,q,m *are integers such that* $p \geq 0$, $q \geq 0$ *and* $m \geq 0$. *Also let* f *be an entire function with relative index-pair* (m,q), *Then*

$$(i)\ \lambda_g^{(p,q)}(f) = \infty \text{ when } \rho_h^{(m,p)}(g) = 0,$$

$$(ii)\ \rho_g^{(p,q)}(f) = \infty \text{ when } \lambda_h^{(m,p)}(g) = 0,$$

$$(iii)\ \lambda_g^{(p,q)}(f) = 0 \text{ when } \rho_h^{(m,p)}(g) = \infty$$

and

$$(iv)\ \rho_g^{(p,q)}(f) = 0 \text{ when } \lambda_h^{(m,p)}(g) = \infty.$$

Corollary 8. *Let f, g and h be any three entire functions represented by vector valued Dirichlet series. Also let p, q, m are integers such that $p \geq 0$, $q \geq 0$ and $m \geq 0$. Also let g be an entire function with relative index-pair (m, p), Then*

$$\begin{aligned} (i)\ \rho_g^{(p,q)}(f) &= 0 \text{ when } \rho_h^{(m,q)}(f) = 0, \\ (ii)\ \lambda_g^{(p,q)}(f) &= 0 \text{ when } \lambda_h^{(m,q)}(f) = 0, \\ (iii)\ \rho_g^{(p,q)}(f) &= \infty \text{ when } \rho_h^{(m,q)}(f) = \infty \end{aligned}$$

and

$$(iv)\ \lambda_g^{(p,q)}(f) = \infty \text{ when } \lambda_h^{(m,q)}(f) = \infty.$$

Remark 1. Under the same conditions of Theorem 1, one may write $\rho_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}$ and $\lambda_g^{(p,q)}(f) = \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}$ when $\lambda_h^{(m,p)}(g) = \rho_h^{(m,p)}(g)$. Similarly $\rho_g^{(p,q)}(f) = \frac{\lambda_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}$ and $\lambda_g^{(p,q)}(f) = \frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}$ when $\lambda_h^{(m,q)}(f) = \rho_h^{(m,q)}(f)$.

Next we prove our theorem based on (p, q)-th relative Ritt type and (p, q)-th relative Ritt weak type of entire functions represented by VVDS.

Theorem 2. *Let f and g be any two entire functions VVDS defined by* (1) *with relative index-pairs (m, q) and (m, p) with respect to another entire function h VVDS defined by* (1) *respectively where p, q, m are integers such that $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then*

$$\max\left\{ \left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)} \right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\Delta_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)} \right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} \right\} \leq$$
$$\Delta_g^{(p,q)}(f) \leq \left(\frac{\Delta_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)} \right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

Proof. From the definitions of $\Delta_h^{(m,q)}(f)$ and $\overline{\Delta}_h^{(m,q)}(f)$, we have for all sufficiently large values of σ that

$$(12)\quad M_f(\sigma) \leq M_h\left(\exp^{[m-1]}\left(\left(\Delta_h^{(m,q)}(f) + \varepsilon \right) \left[\log^{[q-1]} \sigma \right]^{\rho_h^{(m,q)}(f)} \right) \right),$$

$$(13)\quad M_f(\sigma) \geq M_h\left(\exp^{[m-1]}\left(\left(\overline{\Delta}_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right)\right)$$

and also for a sequence of values of σ tending to infinity, we get that

$$(14)\quad M_f(\sigma) \geq M_h\left(\exp^{[m-1]}\left(\left(\Delta_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right)\right),$$

$$(15)\quad M_f(\sigma) \leq M_h\left(\exp^{[m-1]}\left(\left(\overline{\Delta}_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right)\right).$$

Similarly from the definitions of $\Delta_h^{(m,p)}(g)$ and $\overline{\Delta}_h^{(m,p)}(g)$, it follows for all sufficiently large values of σ that

$$M_g(\sigma) \leq M_h\left(\exp^{[m-1]}\left(\left(\Delta_h^{(m,p)}(g)+\varepsilon\right)\left[\log^{[p-1]}\sigma\right]^{\rho_h^{(m,p)}(g)}\right)\right)$$

$$(16)\quad i.e.,\ M_h(\sigma) \geq M_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\sigma}{\left(\Delta_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right) \text{ and}$$

$$(17)\quad M_h(\sigma) \leq M_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\sigma}{\left(\overline{\Delta}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right).$$

Also for a sequence of values of σ tending to infinity, we obtain that

$$(18)\quad M_h(\sigma) \leq M_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\sigma}{\left(\Delta_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right) \text{ and}$$

$$(19)\quad M_h(\sigma) \geq M_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\sigma}{\left(\overline{\Delta}_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right).$$

From the definitions of $\overline{\tau}_h^{(m,q)}(f)$ and $\tau_h^{(m,q)}(f)$, we have for all sufficiently large values of σ that

$$M_f(\sigma) \leq M_h\left(\exp^{[m-1]}\left(\left(\overline{\tau}_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_h^{(m,q)}(f)}\right)\right), \tag{20}$$

$$M_f(\sigma) \geq M_h\left(\exp^{[m-1]}\left(\left(\tau_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_h^{(m,q)}(f)}\right)\right) \tag{21}$$

and also for a sequence of values of σ tending to infinity, we get that

$$M_f(\sigma) \geq M_h\left(\exp^{[m-1]}\left(\left(\overline{\tau}_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_h^{(m,q)}(f)}\right)\right), \tag{22}$$

$$M_f(\sigma) \leq M_h\left(\exp^{[m-1]}\left(\left(\tau_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\lambda_h^{(m,q)}(f)}\right)\right). \tag{23}$$

Similarly from the definitions of $\overline{\tau}_h^{(m,p)}(g)$ and $\tau_h^{(m,p)}(g)$, it follows for all sufficiently large values of σ that

$$M_g(\sigma) \leq M_h\left(\exp^{[m-1]}\left(\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)\left[\log^{[p-1]}\sigma\right]^{\lambda_h^{(m,p)}(g)}\right)\right)$$

$$i.e.,\ M_h(\sigma) \geq M_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\sigma}{\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right) \quad \text{and} \tag{24}$$

$$M_h(\sigma) \leq M_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\sigma}{\left(\tau_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right). \tag{25}$$

Also for a sequence of values of σ tending to infinity, we obtain that

$$M_h(\sigma) \leq M_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\sigma}{\left(\overline{\tau}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right) \quad \text{and} \tag{26}$$

$$M_h(\sigma) \geq M_g\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\sigma}{\left(\tau_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_g(m,p)}}\right). \tag{27}$$

Now from (14) and in view of (24), we get for a sequence of values of σ tending to infinity that

$$M_g^{-1}(M_f(\sigma)) \geq$$
$$M_g^{-1}\left(M_h\left(\exp^{[m-1]}\left(\left(\Delta_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right)\right)\right)$$

$$i.e.,\ M_g^{-1}M_f(\sigma) \geq$$
$$\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left(\left(\Delta_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right)}{\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right)$$

$$i.e.,\ \log^{[p-1]}M_g^{-1}(M_f(\sigma)) \geq \frac{\left(\Delta_h^{(m,q)}(f)-\varepsilon\right)}{\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)}^{\frac{1}{\lambda_h^{(m,p)}(g)}}\cdot\left[\log^{[q-1]}\sigma\right]^{\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}}.$$

Since in view of Theorem 1, $\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} \geq \rho_g^{(p,q)}(f)$ and as $\varepsilon(>0)$ is arbitrary, therefore it follows from above that

$$\varliminf_{\sigma\to\infty}\frac{\log^{[p-1]}M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_g^{(p,q)}(f)}} \geq \left(\frac{\Delta_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

$$i.e.,\ \Delta_g^{(p,q)}(f) \geq \left(\frac{\Delta_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}. \tag{28}$$

Similarly from (13) and in view of (27), it follows for a sequence of values of σ tending to infinity that

$$M_g^{-1}(M_f(\sigma)) \geq$$
$$M_g^{-1}\left(M_h\left(\exp^{[m-1]}\left(\left(\overline{\Delta}_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right)\right)\right)$$

i.e., $M_g^{-1} M_f(\sigma) \geq$

$$\left(\exp^{[p-1]} \left(\frac{\log^{[m-1]} \exp^{[m-1]} \left(\left(\overline{\Delta}_h^{(m,q)}(f) - \varepsilon \right) \left[\log^{[q-1]} \sigma \right]^{\rho_h^{(m,q)}(f)} \right)}{\left(\tau_h^{(m,p)}(g) + \varepsilon \right)} \right)^{\frac{1}{\lambda_g(m,p)}} \right)$$

$$i.e., \ \log^{[p-1]} M_g^{-1}(M_f(\sigma)) \geq \left(\frac{\left(\overline{\Delta}_h^{(m,q)}(f) - \varepsilon \right)}{\left(\tau_h^{(m,p)}(g) + \varepsilon \right)} \right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} \cdot \left[\log^{[q-1]} \sigma \right]^{\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}}.$$

Since in view of Theorem 1, it follows that $\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)} \geq \rho_g^{(p,q)}(f)$. Also $\varepsilon\,(> 0)$ is arbitrary, so we get from above that

$$\varliminf_{\sigma \to \infty} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]} \sigma \right]^{\rho_g^{(p,q)}(f)}} \geq \left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)} \right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

$$i.e., \ \Delta_g^{(p,q)}(f) \geq \left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)} \right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}. \tag{29}$$

Again in view of (17), we have from (12) for all sufficiently large values of σ that

$M_g^{-1}(M_f(\sigma)) \leq$

$$M_g^{-1} \left(M_h \left(\exp^{[m-1]} \left(\left(\Delta_h^{(m,q)}(f) + \varepsilon \right) \left[\log^{[q-1]} \sigma \right]^{\rho_h^{(m,q)}(f)} \right) \right) \right)$$

i.e., $M_g^{-1} M_f(\sigma) \leq$

$$\left(\exp^{[p-1]} \left(\frac{\log^{[m-1]} \exp^{[m-1]} \left(\left(\Delta_h^{(m,q)}(f) + \varepsilon \right) \left[\log^{[q-1]} \sigma \right]^{\rho_h^{(m,q)}(f)} \right)}{\left(\overline{\Delta}_h^{(m,p)}(g) - \varepsilon \right)} \right)^{\frac{1}{\rho_h^{(m,p)}(g)}} \right)$$

$$i.e.,\ \log^{[p-1]} M_g^{-1}\left(M_f\left(\sigma\right)\right) \leq \left(\frac{\left(\Delta_h^{(m,q)}\left(f\right)+\varepsilon\right)}{\left(\overline{\Delta}_h^{(m,p)}\left(g\right)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}} \cdot \left[\log^{[q-1]}\sigma\right]^{\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}}.$$

As in view of Theorem 1, it follows that $\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \rho_g^{(p,q)}\left(f\right)$ Since $\varepsilon\left(>0\right)$ is arbitrary, we get from above that

$$\overline{\lim_{\sigma\to\infty}} \frac{\log^{[p-1]} M_g^{-1}\left(M_f\left(\sigma\right)\right)}{\left[\log^{[q-1]}\sigma\right]^{\rho_g^{(p,q)}(f)}} \leq \left(\frac{\Delta_h^{(m,q)}\left(f\right)}{\overline{\Delta}_h^{(m,p)}\left(g\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

$$(30) \qquad i.e.,\ \Delta_g^{(p,q)}\left(f\right) \leq \left(\frac{\Delta_h^{(m,q)}\left(f\right)}{\overline{\Delta}_h^{(m,p)}\left(g\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

Thus the theorem follows from (28), (29) and (30). □

The conclusion of the following corollary can be carried out from (17) and (20); (20) and (25) respectively after applying the same technique of Theorem 2 and with the help of Theorem 1. Therefore its proof is omitted.

Corollary 9. *Let f and g be any two entire functions VVDS defined by* (1) *with relative index-pairs* (m, q) *and* (m, p) *with respect to another entire function h VVDS defined by* (1) *respectively where p, q, m are integers such that $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then*

$$\Delta_g^{(p,q)}\left(f\right) \leq \min\left\{\left(\frac{\overline{\tau}_h^{(m,q)}\left(f\right)}{\tau_h^{(m,p)}\left(g\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}\left(f\right)}{\overline{\Delta}_h^{(m,p)}\left(g\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}.$$

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carried out the following theorem from pairwise inequalities numbers (21) and (24) ; (18) and (20); (17) and (23) respectively and therefore its proofs is omitted:

Theorem 3. *Let f and g be any two entire functions VVDS defined by* (1) *with relative index-pairs* (m, q) *and* (m, p) *with respect to another entire function h VVDS defined by* (1) *respectively where p, q, m are* *integers such that* $p \geq 0$, $q \geq 0$ *and* $m \geq 0$. *Then*

$$\left(\frac{\tau_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)} \right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} \leq \tau_g^{(p,q)}(f) \leq \min \left\{ \left(\frac{\tau_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)} \right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)} \right)^{\frac{1}{\rho_h^{(m,p)}(g)}} \right\}.$$

Corollary 10. *Let* f *and* g *be any two entire functions VVDS defined by* (1) *with relative index-pairs* (m,q) *and* (m,p) *with respect to another entire function* h *VVDS defined by* (1) *respectively where* p, q, m *are integers such that* $p \geq 0$, $q \geq 0$ *and* $m \geq 0$. *Then*

$$\tau_g^{(p,q)}(f) \geq \max \left\{ \left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)} \right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)} \right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} \right\}.$$

With the help of Theorem 1, the conclusion of the above corollary can be carry out from (13) , (16) and (13) , (24) respectively after applying the same technique of Theorem 2 and therefore its proof is omitted.

Theorem 4. *Let* f *and* g *be any two entire functions VVDS defined by* (1) *with relative index-pairs* (m,q) *and* (m,p) *with respect to another entire function* h *VVDS defined by* (1) *respectively where* p, q, m *are integers such that* $p \geq 0$, $q \geq 0$ *and* $m \geq 0$. *Then*

$$\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)} \right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} \leq \overline{\Delta}_g^{(p,q)}(f) \leq \min \left\{ \left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)} \right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\Delta_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)} \right)^{\frac{1}{\rho_h^{(m,p)}(g)}} \right\}.$$

Proof. From (13) and in view of (24), we get for all sufficiently large values of σ that

$$M_g^{-1}(M_f(\sigma)) \geq M_g^{-1}\left(M_h \left(\exp^{[m-1]} \left(\left(\overline{\Delta}_h^{(m,q)}(f) - \varepsilon \right) \left[\log^{[q-1]} \sigma \right]^{\rho_h^{(m,q)}(f)} \right) \right) \right)$$

i.e., $M_g^{-1} M_f(\sigma) \geq$

$$\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left(\left(\overline{\Delta}_h^{(m,q)}(f)-\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right)}{\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right)$$

$$i.e.,\ \log^{[p-1]}M_g^{-1}(M_f(\sigma))\geq\left(\frac{\left(\overline{\Delta}_h^{(m,q)}(f)-\varepsilon\right)}{\left(\overline{\tau}_h^{(m,p)}(g)+\varepsilon\right)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\cdot\left[\log^{[q-1]}\sigma\right]^{\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}}.$$

Now in view of Theorem 1, it follows that $\frac{\rho_h^{(m,q)}(f)}{\lambda_h^{(m,p)}(g)}\geq\rho_g^{(p,q)}(f)$. Since $\varepsilon(>0)$ is arbitrary, we get from above that

$$\varliminf_{\sigma\to\infty}\frac{\log^{[p-1]}M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_g^{(p,q)}(f)}}\geq\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}$$

$$i.e.,\ \overline{\Delta}_g^{(p,q)}(f)\geq\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}. \tag{31}$$

Further in view of (18), we get from (12) for a sequence of values of σ tending to infinity that

$$M_g^{-1}(M_f(\sigma))\leq$$
$$M_g^{-1}\left(M_h\left(\exp^{[m-1]}\left[\left(\Delta_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right]\right)\right)$$

$$i.e.,\ M_g^{-1}M_f(\sigma)\leq$$
$$\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left(\left(\Delta_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right)}{\left(\Delta_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right)$$

$$i.e.,\ \log^{[p-1]}M_g^{-1}(M_f(\sigma))\leq\left(\frac{\left(\Delta_h^{(m,q)}(f)+\varepsilon\right)}{\left(\Delta_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\cdot\left[\log^{[q-1]}\sigma\right]^{\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}}.$$

Again as in view of Theorem 1, $\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \rho_g^{(p,q)}(f)$ and $\varepsilon\,(>0)$ is arbitrary, therefore we get from above that

$$\varlimsup_{\sigma\to\infty} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_g^{(p,q)}(f)}} \leq \left(\frac{\Delta_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

$$\text{(32)} \qquad i.e.,\ \overline{\Delta}_g^{(p,q)}(f) \leq \left(\frac{\Delta_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

Likewise from (15) and in view of (17), it follows for a sequence of values of σ tending to infinity that

$$M_g^{-1}(M_f(\sigma)) \leq$$

$$M_g^{-1}\left(M_h\left(\exp^{[m-1]}\left(\left(\overline{\Delta}_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right)\right)\right)$$

$$i.e.,\ M_g^{-1}M_f(\sigma) \leq$$

$$\left(\exp^{[p-1]}\left(\frac{\log^{[m-1]}\exp^{[m-1]}\left(\left(\overline{\Delta}_h^{(m,q)}(f)+\varepsilon\right)\left[\log^{[q-1]}\sigma\right]^{\rho_h^{(m,q)}(f)}\right)}{\left(\overline{\Delta}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right)$$

$$i.e.,\ \log^{[p-1]} M_g^{-1}(M_f(\sigma)) \leq$$

$$\text{(33)} \qquad \left(\frac{\left(\overline{\Delta}_h^{(m,q)}(f)+\varepsilon\right)}{\left(\overline{\Delta}_h^{(m,p)}(g)-\varepsilon\right)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}} \cdot \left[\log^{[q-1]}\sigma\right]^{\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)}}.$$

Analogously, we get from (33) that

$$\varlimsup_{\sigma\to\infty} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left[\log^{[q-1]}\sigma\right]^{\rho_g^{(p,q)}(f)}} \leq \left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

$$\text{(34)} \qquad i.e.,\ \overline{\Delta}_g^{(p,q)}(f) \leq \left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}},$$

since in view of Theorem 1, $\frac{\rho_h^{(m,q)}(f)}{\rho_h^{(m,p)}(g)} \leq \rho_g^{(p,q)}(f)$ and $\varepsilon\,(>0)$ is arbitrary.

Thus the theorem follows from (31), (32) and (34). □

Corollary 11. *Let f and g be any two entire functions VVDS defined by* (1) *with relative index-pairs* (m,q) *and* (m,p) *with respect to another entire function h VVDS defined by* (1) *respectively where p,q,m are integers such that $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then*

$$\overline{\Delta}_g^{(p,q)}(f) \leq \min\left\{ \left(\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\sigma_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\tau_h^{(m,q)}(f)}{\overline{\sigma}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}} \right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities no (17) and (23); (18) and (20); (23) and (25); (20) and (26) respectively after applying the same technique of Theorem 4 and with the help of Theorem 1. Therefore its proof is omitted.

Similarly in the line of Theorem 2 and with the help of Theorem 1, one may easily carried out the following theorem from pairwise inequalities no (22) and (24); (21) and (27); (17) and (20) respectively and therefore its proofs is omitted:

Theorem 5. *Let f and g be any two entire functions VVDS defined by* (1) *with relative index-pairs* (m,q) *and* (m,p) *with respect to another entire function h VVDS defined by* (1) *respectively where p,q,m are integers such that $p \geq 0$, $q \geq 0$ and $m \geq 0$. Then*

$$\max\left\{ \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_g(m,p)}}, \left(\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} \right\} \leq \overline{\tau}_g^{(p,q)}(f) \leq \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}.$$

Corollary 12. *Let f and g be any two entire functions VVDS defined by* (1) *with relative index-pairs* (m,q) *and* (m,p) *with respect to another entire function h VVDS defined by* (1) *respectively where*

p,q,m are integers such that $p\geq 0$, $q\geq 0$ and $m\geq 0$. Then

$$\overline{\tau}_g^{(p,q)}(f)\geq \max\left\{\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}},\left(\frac{\Delta_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}},\right.$$

$$\left.\left(\frac{\Delta_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}},\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}.$$

The conclusion of the above corollary can be carried out from pairwise inequalities no (14) and (16) ; (13) and (19) ; (14) and (24); (13) and (27) respectively after applying the same technique of Theorem 4 and with the help of Theorem 1. Therefore its proof is omitted.

Now we state the following two theorems without their proofs as because they can be derived easily using the same technique or with some easy reasoning by the help of with the help of Remark 1 and therefore left to the readers.

Theorem 6. *Let f and g be any two entire functions VVDS defined by* (1) *with relative index-pairs* (m,q) *and* (m,p) *with respect to another entire function h VVDS defined by* (1) *respectively where p,q,m are integers such that $p\geq 0$, $q\geq 0$ and $m\geq 0$. Also let $\lambda_h^{(m,p)}(g)=\rho_h^{(m,p)}(g)$. Then*

$$\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\leq \overline{\Delta}_g^{(p,q)}(f)$$

$$\leq \min\left\{\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}},\left(\frac{\Delta_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}$$

$$\leq \max\left\{\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}},\left(\frac{\Delta_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}$$

$$\leq \Delta_g^{(p,q)}(f)\leq\left(\frac{\Delta_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

and

$$\left(\frac{\tau_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} \leq \tau_g^{(p,q)}(f)$$
$$\leq \min\left\{\left(\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}$$
$$\leq \max\left\{\left(\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}$$
$$\leq \overline{\tau}_g^{(p,q)}(f) \leq \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}.$$

Theorem 7. *Let f and g be any two entire functions VVDS defined by (1) with relative index-pairs (m,q) and (m,p) with respect to another entire function h VVDS defined by (1) respectively where p,q,m are integers such that $p \geq 0$, $q \geq 0$ and $m \geq 0$. Also let $\lambda_h^{(m,q)}(f) = \rho_h^{(m,q)}(f)$. Then*

$$\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}} \leq \tau_g^{(p,q)}(f)$$
$$\leq \min\left\{\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\Delta_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}$$
$$\leq \max\left\{\left(\frac{\overline{\Delta}_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}, \left(\frac{\Delta_h^{(m,q)}(f)}{\Delta_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}\right\}$$
$$\leq \overline{\tau}_g^{(p,q)}(f) \leq \left(\frac{\Delta_h^{(m,q)}(f)}{\overline{\Delta}_h^{(m,p)}(g)}\right)^{\frac{1}{\rho_h^{(m,p)}(g)}}$$

and

$$\left(\frac{\tau_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}} \leq \overline{\Delta}_g^{(p,q)}(f)$$
$$\leq \min\left\{\left(\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}$$
$$\leq \max\left\{\left(\frac{\tau_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}, \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\overline{\tau}_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}\right\}$$
$$\leq \Delta_g^{(p,q)}(f) \leq \left(\frac{\overline{\tau}_h^{(m,q)}(f)}{\tau_h^{(m,p)}(g)}\right)^{\frac{1}{\lambda_h^{(m,p)}(g)}}.$$

3. Conclusion

The main aim of the present paper is to revisit some growth properties of entire functions in the form of vector valued Dirichlet series on the basis of their (p, q)-th relative Ritt order, (p, q)-th relative Ritt type and (p, q)-th relative Ritt weak type for any two positive integers p and q. Recently, Filevych et al. [7] and Glova et al. [8] have studied the generalizations of the growth properties of Dirichlet series. Further, Hlova et al. [9] have investigated some problems regarding the generalized types of the growth of Dirichlet series. The notion involved in our paper may be reinvestigated in the light of the theories employed in [7], [8] and [9].

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