PSEUDO-METRIC ON KU-ALGEBRAS

ALI N.A. KOAM, AZEEM HAIDER, AND MOIN A. ANSARI*

ABSTRACT. In this paper we have introduced the concept of pseudo-metric which we induced from a pseudo-valuation on KU-algebras and investigated the relationship between pseudo-valuations and ideals of KU-algebras. Conditions for a real-valued function to be a pseudo-valuation on KU-algebras are provided.

1. Introduction

Pseudo-metric induce by pseudo-valuations on Hilbert algebras was initially introduced by Busnæag [2]. Further Busnæag [3] proved many results on extensions of pseudo-valuations. Pseudo-valuations in residuated lattices was introduced by Busnæag [4] where many theorems based on pseudo-valuations in lattice terms and their extension for residuated lattices to pseudo-valuation from valuations has been shown using the model of Hilbert algebras [3].

Logical algebras have become the keen interest for researchers in recent years and intensively studied under the influence of different mathematical concepts. Doh and Kang [5] introduced the concept of pseudo-valuation on BCK/BCI algebras and studied results based on them. Ghorbani [6] defined congruence relations and gave quotient structure


We define a pseudo-valuations on KU-algebras using the model of Busnag and introduce a pseudo-metric on KU-algebras. We also prove that the binary operation defined on KU-algebras is uniformly continuous under the induced pseudo-metric.

2. Preliminaries

In this section, we shall consider concepts based on KU-algebras, KU-ideals and other important terminologies with examples and some related results.

**Definition 1.** [8] By a KU-algebra we mean an algebra $(X, \circ, 1)$ of type $(2,0)$ with a single binary operation $\circ$ that satisfies the following identities: for any $x, y, z \in X$,

- $(ku1)$ $(x \circ y) \circ [(y \circ z) \circ (x \circ z)] = 1$,
- $(ku2)$ $x \circ 1 = 1$,
- $(ku3)$ $1 \circ x = x$,
- $(ku4)$ $x \circ y = y \circ x = 1$ implies $x = y$.

In what follows, let $(X, \circ, 1)$ denote a KU-algebra unless otherwise specified. For brevity we also call $X$ a KU-algebra. In $X$ we can define a binary relation $\leq$ by : $x \leq y$ if and only if $x \circ y = 1$.

**Lemma 1.** [8] $(X, \circ, 1)$ is a KU-algebra if and only if it satisfies:

- $(ku5)$ $x \circ y \leq (y \circ z) \circ (x \circ z)$,
- $(ku6)$ $x \leq 1$,
- $(ku7)$ $x \leq y, y \leq x$ implies $x = y$.

**Lemma 2.** In a KU-algebra, the following identities are true [7]:

- (1) $z \circ z = 1$,
- (2) $z \circ (x \circ z) = 1$,
(3) \( x \leq y \) imply \( y \circ z \leq x \circ z \),
(4) \( z \circ (y \circ x) = y \circ (z \circ x) \),
(5) \( y \circ [(y \circ x) \circ x] = 1 \), for all \( x, y, z \in X \).

**Example 1.** [7] Let \( X = \{1, 2, 3, 4, 5\} \) in which \( \circ \) is defined by the following table

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It is easy to see that \( X \) is a KU-algebra.

**Definition 2.** [8] A non-empty subset \( A \) of a KU-algebra \( X \) is called a KU-ideal of \( X \) if it satisfies the following conditions:

(1) \( 1 \in A \),
(2) \( x \circ (y \circ z) \in A \), \( y \in A \) imply \( x \circ z \in A \), for all \( x, y, z \in X \).

**Example 2.** [1] Let \( X = \{1, 2, 3, 4, 5, 6\} \) in which \( \circ \) is defined by the following table:

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Clearly \( (X, \circ, 1) \) is a KU-algebra. It is easy to show that \( A = \{1, 2\} \) and \( B = \{1, 2, 3, 4, 5\} \) are KU-ideals of \( X \).

### 3. Pseudo-valuations on KU-algebras

**Definition 3.** A real-valued function \( \zeta \) on a KU-algebra \( X \) is called a pseudo-valuation on \( X \) if it satisfies the following two conditions:

(1) \( \zeta(1) = 0 \)
(2) \( \zeta(x \circ z) \leq \zeta(x \circ (y \circ z)) + \zeta(y) \forall x, y, z \in X \)

A pseudo-valuation \( \zeta \) on a KU-algebra \( X \) satisfying the following condition:
\[ \zeta(x) = 0 \Rightarrow x = 1 \ \forall x \in X \] is called a valuation on \( X \).

**Example 3.** Let \( X = \{1, 2, 3, 4\} \) be a set with operation \( \circ \). A table for such \( X \) is defined by following table

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Here \( X \) is a KU-algebra. We find that a real valued function defined on \( X \) by

\[ \zeta(1) = 0, \ zeta(2) = 1, \ zeta(3) = \zeta(4) = 3, \] is a pseudo-valuation on \( X \).

**Proposition 1.** Let \( \zeta \) be a pseudo-valuation on a KU-algebra \( X \). Then we have

1. \( x \leq y \Rightarrow \zeta(y) \leq \zeta(x) \).
2. \( \zeta(x \circ y) \leq \zeta(y) \ \forall x, y \in X \).
3. \( \zeta((x \circ (y \circ z)) \circ z) \leq \zeta(x) + \zeta(y) \ \forall x, y, z \in X \).

**Proof.** (1) Let \( x, y \in X \) be such that \( x \leq y \). Now choosing \( x = 1, y = x, z = y \), in Definition 3(1),(2) and using (ku3) we get

\[ \zeta(y) = \zeta(1 \circ y) \leq \zeta(1 \circ (x \circ y)) + \zeta(x) = \zeta(1) + \zeta(x) = \zeta(x) \]

(2) If we choose \( z = y \) in Definition 3(2), then we get \( \zeta(x \circ y) \leq \zeta(x \circ (y \circ y)) + \zeta(y) = \zeta(x \circ 1) + \zeta(y) = \zeta(1) + \zeta(y) = \zeta(y) \ \forall x, y \in X \).

(3) If we choose \( x = x \circ (y \circ z) \) in Definition 3(2) then we get

\[ \zeta((x \circ (y \circ z)) \circ z) \leq \zeta((x \circ (y \circ z)) \circ (y \circ z)) + \zeta(y) \]

Now using the relation \( \leq \) and Lemma 2 (5), we get \( x \leq (x \circ (y \circ z)) \circ (y \circ z) \). By Proposition 1, it follows that \( \zeta((x \circ (y \circ z)) \circ (y \circ z)) \leq \zeta(x) \) using this relation in Equation 3.1, we get \( \zeta((x \circ (y \circ z)) \circ z) \leq \zeta(x) + \zeta(y) \ \forall x, y, z \in X \).

**Corollary 1.** Every pseudo-valuation \( \zeta \) on a KU-algebra \( X \) satisfies the following inequality \( \zeta(x) \geq 0 \ \forall x \in X \).

**Proposition 2.** If \( \zeta \) is a pseudo-valuation on a KU-algebra \( X \), then we have

\[ \zeta((x \circ y) \circ y) \leq \zeta(x) \ \forall x, y \in X. \]
Proof. Choosing \( y = 1 \) and \( z = y \) in Proposition 1, using (ku3) and Definition 3(1) we get that
\[
\zeta((x \circ y) \circ y) = \zeta((x \circ (1 \circ y)) \circ y) \leq \zeta(x) + \zeta(1) = \zeta(x) \forall x, y \in X. \quad \square
\]

The following theorem provides conditions for a real valued function on a KU-algebra \( X \) to be a pseudo-valuation on \( X \).

**Theorem 1.** Let \( \zeta \) be a real valued function on a KU-algebra \( X \) satisfying the following conditions.
(1) If \( \zeta(a) \leq \zeta(x) \forall x \in X \), then \( \zeta(a) = 1 \).
(2) \( \zeta(x \circ y) \leq \zeta(y) \forall x, y \in X \).
(3) \( \zeta((x \circ (y \circ z)) \circ z) \leq \zeta(x) + \zeta(y) \).

Then \( \zeta \) is a pseudo-valuation on \( X \).

**Proof.** From Lemma 2 (1) and given condition (2), we have \( \zeta(1) = \zeta(x \circ x) \leq \zeta(x) \forall x \in X \) and hence \( \zeta(1) = 0 \), using given condition (1).

Now, from (ku3), Lemma 2 (1) and given condition (3), we get \( \zeta(y) = \zeta((x \circ y) \circ (x \circ y)) \circ y \leq \zeta(x \circ y) + \zeta(x) \forall x, y \in X \). It follows from Lemma 2 (4) that \( \zeta(x \circ z) \leq \zeta(y \circ (x \circ z)) + \zeta(y) = \zeta(x \circ (y \circ z)) + \zeta(y) \forall x, y, z \in X \).

Therefore \( \zeta \) is a pseudo-valuation on \( X \). \( \square \)

**Corollary 2.** Let \( \zeta \) be a real-valued function on a KU-algebra \( X \) satisfying the following conditions:
(1) \( \zeta(1) = 0 \)
(2) \( \zeta(x \circ y) \leq \zeta(y) , \forall x, y \in X \).
(3) \( \zeta((x \circ (y \circ z)) \circ z) \leq \zeta(x) + \zeta(y) , \forall x, y, z \in X \).

Then \( \zeta \) is a pseudo-valuation on \( X \).

**Theorem 2.** If \( \zeta \) is a pseudo-valuation on a KU-algebra \( X \), then \( \zeta(y) \leq \zeta(x \circ y) + \zeta(x) , \forall x, y \in X \).

**Proof.** Let \( m = (x \circ y) \circ y \) for any \( x, y \in X \), and \( n = x \circ y \).

Then \( y = 1 \circ y = ((x \circ y) \circ y) \circ y = (m \circ (n \circ y)) \circ y \). It follows from Theorem 2, Propositions 1 and Propositions 2 that \( \zeta(y) = \zeta(m \circ (n \circ y)) \circ y) \leq \zeta(m) + \zeta(n) = \zeta((x \circ y) \circ y) + \zeta(x \circ y) \leq \zeta(x) + \zeta(x \circ y) \). This completes the proof. \( \square \)

**Theorem 3.** Let \( \zeta \) be a real-valued function on a KU-algebra \( X \) satisfying the following conditions.
(1) \( \zeta(1) = 0 \)
(2) \( \zeta(y) \leq \zeta(x \circ y) + \zeta(x) , \forall x, y \in X \).

Then \( \zeta \) is a pseudo-valuation on \( X \).
Proof. By Lemma 2 (4), Lemma 2 (5) and given condition (2), we have
\[\zeta([b \circ (a \circ a) \circ x]) \leq \zeta(b \circ ((b \circ (a \circ x)) \circ x) + \zeta(b) \text{ (by given condition (2)}) \]
\[\leq \zeta([b \circ (a \circ x)) \circ (b \circ x)] + \zeta(b) \text{ (by Lemma 2 (4))} \]
\[= \zeta((a \circ (b \circ x)) \circ (b \circ x)) + \zeta(a) + \zeta(b) \text{ (by given condition (2))} \]
\[= \zeta(a) + \zeta(b) \text{ (by Lemma 2(5)).} \]

Also \(\zeta(x \circ y) \leq \zeta(y)\) by Lemma 2(2) and Proposition 1(1). Using Corollary 2 we get that \(\zeta\) is a pseudo-valuation on \(X\).

PROPOSITION 3. If \(\zeta\) is a pseudo-valuation on a KU-algebra \(X\), then
\[(3.2) \quad a \leq b \circ x \Rightarrow \zeta(x) \leq \zeta(a) + \zeta(b) \quad \forall a, b, x \in X.\]

Proof. Suppose that \(a, b, x \in X\) such that \(a \leq b \circ x\). Then by Proposition 1 (3) and Theorem 2, we have that
\[\zeta(x) \leq \zeta((a \circ (b \circ x)) \circ x) + \zeta(a) + \zeta(b) \text{ (by given condition (2))} \]
\[= \zeta((a \circ (b \circ x)) \circ x) + \zeta(a) + \zeta(b) \text{ (by Lemma 2(5))} \]
\[= \zeta(a) + \zeta(b). \]

THEOREM 4. Let \(\zeta\) be a real-valued function on a KU-algebra \(X\). If \(\zeta\) satisfies \(\zeta(1) = 0\) and condition (3.2), then \(\zeta\) is a pseudo-valuation on \(X\).

Proof. From Lemma 2 (5), we have \(a \circ ((a \circ x) \circ x) = 1\), which implies from \(x \leq y \iff x \circ y = 1\) that \(a \leq (a \circ x) \circ x, \forall a, x \in X\). Thus it follows from Proposition 3 that \(\zeta(x) \leq \zeta(a \circ x) + \zeta(a), \forall a, x \in X\). Hence from Theorem 3, we conclude that \(\zeta\) is a pseudo-valuation on \(X\). \(\square\)

PROPOSITION 4. Suppose that \(X\) is a KU-algebra. Then every pseudo-valuation \(\zeta\) on \(X\) satisfies the following inequality:
\[\zeta(x \circ z) \leq \zeta(x \circ y) + \zeta(y \circ z), \forall x, y, z \in X.\]

Proof. It follows from (ku1) and Theorem 4. \(\square\)

THEOREM 5. If \(\zeta\) is a pseudo-valuation on a KU-algebra \(X\), then the set \(I := \{x \in X | \zeta(x) = 0\}\) is an ideal of \(X\).

Proof. We have \(\zeta(1) = 0\) and hence \(1 \in I\). Next \(x, y, z \in X\) be such that \(y \in I\) and \(x \circ (y \circ z) \in I\). Then \(\zeta(y) = 0\) and \(\zeta(x \circ (y \circ z)) = 0\). By Definition 3(2) we get that \(\zeta(x \circ z) \leq \zeta(x \circ (y \circ z)) + \zeta(y) = 0\) so that \(\zeta(x \circ z) = 0\). Hence \(x \circ z \in I\), and therefore \(I\) is an ideal of \(X\). \(\square\)
Example 4. Let \( X = \{1, 2, 3, 4, 5, 6\} \) in which \( \circ \) is defined by the following table:

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\circ & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 \\
2 & 1 & 1 & 2 & 4 & 4 & 5 \\
3 & 1 & 1 & 1 & 4 & 4 & 4 \\
4 & 1 & 2 & 3 & 1 & 2 & 3 \\
5 & 1 & 1 & 2 & 1 & 1 & 2 \\
6 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Clearly \( X \) is a KU-algebra. Now define a real-valued function \( \zeta \) on \( X \) by \( \zeta(1) = \zeta(2) = \zeta(3) = 0, \zeta(4) = 3, \zeta(5) = 1 \) and \( \zeta(6) = 2 \). Then \( I := \{x \in X \mid \zeta(x) = 1\} = \{2, 3, 4\} \) is the ideal of \( X \). But \( \zeta \) is not a pseudo-valuation as \( \zeta(3 \circ 5) \nleq \zeta(3 \circ (5 \circ 5)) + \zeta(5) \).

4. Pseudo-metric on KU-algebras

In this section we define pseudo-metric on KU-algebras and show related results.

Theorem 6. Let \( X \) be a KU-algebra and \( \zeta \) be a pseudo-valuation on \( X \). Then the mapping \( d_\zeta : X \times X \to \mathbb{R} \) defined by \( d_\zeta(x, y) = \zeta(x \circ y) + \zeta(y \circ x) \) \( \forall (x, y) \in X \times X \) is a metric on \( X \), called pseudo-metric induced by pseudo-valuation \( \zeta \) and correspondingly \( (X, d_\zeta) \) is called a pseudo-metric space.

Proof. Clearly, \( d_\zeta(x, y) \geq 1, d_\zeta(x, x) = 1 \) and \( d_\zeta(x, y) = d_\zeta(y, x) \) \( \forall x, y \in X \). For any \( x, y, z \in X \) from Proposition 4, we get that \( d_\zeta(x, y) + d_\zeta(y, z) = [\zeta(x \circ y) + \zeta(y \circ x)] + [\zeta(y \circ z) + \zeta(z \circ y)] = [\zeta(x \circ y) + \zeta(y \circ z)] + [\zeta(z \circ y) + \zeta(y \circ x)] \geq \zeta(x \circ z) + \zeta(z \circ x) = d_\zeta(x, z) \). Hence \( (X, d_\zeta) \) is a pseudo-metric space.

Proposition 5. Let \( X \) be a KU-algebra. Then every pseudo-metric \( d_\zeta \) induced by pseudo-valuation \( \zeta \) satisfies the following inequalities:

1. \( d_\zeta(x, y) \geq d_\zeta(x \circ a, y \circ a) \)
2. \( d_\zeta(x, y) \geq d_\zeta(a \circ x, a \circ y) \),
3. \( d_\zeta(x \circ y, a \circ b) \leq d_\zeta(x \circ y, a \circ y) + d_\zeta(a \circ y, a \circ b) \) \( \forall x, y, a, b \in X \).

Proof. Let \( x, y, a \in X \). By \( (ku5) \) \( x \circ y \leq (y \circ a) \circ (x \circ a) \) and \( y \circ x \leq (x \circ a) \circ (y \circ a) \). It follows from Proposition 1(1) that \( \zeta(x \circ y) \geq \zeta((y \circ a) \circ (x \circ a)) \).
and \( \zeta(y \circ x) \geq \zeta((x \circ a) \circ (y \circ a)) \) so that \( d_\zeta(x, y) = \zeta(x \circ y) + \zeta(y \circ x) \geq \zeta((y \circ a) \circ (x \circ a)) + \zeta((x \circ a) \circ (y \circ a)) = d_\zeta(x \circ a, y \circ a). \)

(2) Similar and followed by proof (1).

(3) Followed by definition of pseudo-metric.

**Theorem 7.** Let \( \zeta \) be a real-valued function on a KU-algebra \( X \), if \( d_\zeta \) is a pseudo-metric on \( X \), then \((X \times X, d_\zeta^2)\) is a pseudo-metric space, where \( d_\zeta^2((x, y), (a, b)) = \max\{d_\zeta(x, a), d_\zeta(y, b)\} \ \forall (x, y), (a, b) \in X \times X. \)

**Proof.** Suppose \( d_\zeta \) is a pseudo-metric on \( X \). For any \((x, y), (a, b) \in X \times X\), we have \( d_\zeta^2((x, y), (x, y)) = \max\{d_\zeta(x, x), d_\zeta(y, y)\} = 0 \) and \( d_\zeta^2((x, y), (a, b)) = \max\{d_\zeta(x, a), d_\zeta(y, b)\} = \max\{d_\zeta(a, x), d_\zeta(b, y)\} = d_\zeta^2((a, b), (x, y)). \)

Now let \((x, y), (a, b), (u, v) \in X \times X\). Then we have \( d_\zeta^2((x, y), (u, v)) + d_\zeta^2((u, v), (a, b)) = \max\{d_\zeta(x, u), d_\zeta(y, v)\} + \max\{d_\zeta(u, a), d_\zeta(v, b)\} \geq \max\{d_\zeta(x, u) + d_\zeta(y, v) + d_\zeta(u, a) + d_\zeta(v, b)\} \geq \max\{d_\zeta(x, a), d_\zeta(y, b)\} = d_\zeta^2((x, y), (a, b)). \)

Hence \((X \times X, d_\zeta^2)\) is a pseudo-metric space. \( \square \)

**Corollary 3.** If \( \zeta : X \to \mathbb{R} \) is a pseudo-valuation on a KU-algebra \( X \), then \((X \times X, d_\zeta^2)\) is a pseudo-metric space.

**Theorem 8.** Let \( X \) be a KU-algebra. Further if \( \zeta : X \to \mathbb{R} \) is a valuation on \( X \), then \((X \times X, d_\zeta)\) is a metric space.

**Proof.** Suppose \( \zeta \) is a valuation on \( X \). Then \((X, d_\zeta)\) is a pseudo-metric space by Theorem 6. Further consider \( x, y \in X \) be such that \( d_\zeta(x, y) = 0 \). Then \( 0 = d_\zeta(x, y) = \zeta(x \circ y) + \zeta(y \circ x) \), and hence \( \zeta(x \circ y) = 0 \) and \( \zeta(y \circ x) = 0 \) since \( \zeta(x) \geq 0 \ \forall x \in X \). And, since \( \zeta \) is a valuation on \( X \), it follows that \( x \circ y = 1 \) and \( y \circ x = 1 \) so from condition in the given theorem that \( x = y \). Hence \((X, d_\zeta)\) is a metric space. \( \square \)

**Theorem 9.** Let \( X \) be a KU-algebra. If \( \zeta : X \to \mathbb{R} \) is a valuation on \( X \), then \((X \times X, d_\zeta^2)\) is a metric space.

**Proof.** From Corollary 3, we have that \((X \times X, d_\zeta^2)\) is a pseudo-metric space. Suppose that \((x, y), (a, b) \in X \times X \) be such that \( d_\zeta^2((x, y), (a, b)) = 0 \). Then \( 0 = d_\zeta^2((x, y), (a, b)) = \max\{d_\zeta(x, a), d_\zeta(y, b)\} \), and so \( d_\zeta(x, a) = 0 = d_\zeta(y, b) \) since \( d_\zeta(x, y) \geq 0 \ \forall (x, y) \in X \times X \). Hence \( 0 = d_\zeta(x, a) = \zeta(x \circ a) + \zeta(a \circ x) \) and \( 0 = d_\zeta(y, b) = \zeta(y \circ b) + \zeta(b \circ y) \). It follows that \( \zeta(x \circ a) = 0 = \zeta(a \circ x) \) and \( \zeta(y \circ b) = 0 = \zeta(b \circ y) \) so that \( x \circ a = 1 = a \circ x \).
and \( y \circ b = 0 = b \circ y \). Now we have \( a = x \) and \( b = y \), and so \( (x, y) = (a, b) \). Therefore \((X \times X, d_\zeta)\) is a metric space. \( \square \)

**Theorem 10.** Let \( X \) be a KU-algebra. If \( \zeta \) is a valuation on \( X \), then the operation \( \circ \) in \( X \) is uniformly continuous.

**Proof.** Consider for any \( \delta > 0 \), if \( d_\zeta((x, y), (a, b)) < \frac{\delta}{2} \) then \( d_\zeta(x, a) < \frac{\delta}{2} \) and \( d_\zeta(y, b) < \frac{\delta}{2} \). This implies that \( d_\zeta(x \circ y, a \circ b) \leq d_\zeta(x \circ y, a \circ y) + d_\zeta(a \circ y, a \circ b) \leq d_\zeta(x, a) + d_\zeta(y, b) < \frac{\delta}{2} + \frac{\delta}{2} = \delta \) (from Proposition 5). Therefore the operation \( \circ : X \times X \to X \) is uniformly continuous. \( \square \)

**References**


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