Korean J. Math. **27** (2019), No. 1, pp. 141–150
https://doi.org/10.11568/kjm.2019.27.1.141

ON A RING PROPERTY RELATED TO NILRADICALS

Hai-lan Jin, Zhelin Piao*, and Sang Jo Yun

Abstract. In this article we investigate the structure of rings in which lower nilradicals coincide with upper nilradicals. Such rings shall be said to be *quasi-2-primal*. It is shown first that the Köthe's conjecture holds for quasi-2-primal rings. So the results in this article may provide interesting and useful information to the study of nilradicals in various situations. In the procedure we study the structure of quasi-2-primal rings, and observe various kinds of quasi-2-primal rings which do roles in ring theory.

1. Basic properties of quasi-2-primal rings

The Köthe's conjecture implies that if a ring has no nonzero nil ideals then it has no nonzero nil one-sided ideals. For more than 90 years significant progress has been made to answer this, however it is still open. In this article we consider a ring property under which the Köthe's conjecture holds. Via this work, we may give useful information to the study related to nilradicals of polynomial rings, matrix rings and factor rings.

Throughout this note every ring is associative with identity unless otherwise stated. Let R be a ring. We write $N_*(R)$, $N^*(R)$, $N(R)$, and $J(R)$ to denote the lower nilradical (i.e., the intersection of all prime

Received December 19, 2018. Revised February 11, 2019. Accepted February 13, 2019.

2010 Mathematics Subject Classification: 16N40, 16N20, 16S36, 16S50.

Key words and phrases: equal-nilradical ring, Köthe's conjecture, lower nilradical, upper nilradical, Jacobson radical, polynomial ring, matrix ring, factor ring.

$*$ Corresponding author.

ideals), the upper nilradical (i.e., the sum of all nil ideals), the set of all nilpotent elements, and the Jacobson radical of R, respectively. Note $N_0(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$ and $N^*(R) \subseteq J(R)$. In this note we concentrate on the case of $N_*(R) = N^*(R)$ and study the structure of rings satisfying $N_*(R) = N^*(R)$.

Given a ring R, we use $R[x]$ (resp. $R[[x]]$) to denote the polynomial (resp. power series) ring with an indeterminate x over R. For $f(x) \in R[x]$, let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. Denote the n by n full matrix ring over R by $Mat_n(R)$, and the n by n upper (resp., lower) triangular matrix ring over R by $U_n(R)$ (resp., $L_n(R)$). Write $D_n(R) = \{(a_{ij}) \in U_n(R) \mid a_{11} = \cdots = a_{nn}\}$. Use E_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n).

A ring is usually called *reduced* if it has no nonzero nilpotent elements. We first recall two kinds of generalizations of commutative rings. Following Birkenmeier et al. [4], a ring R is called 2*-primal* if $N_*(R) = N(R)$. It is obvious that R is 2-primal if and only if $R/N_*(R)$ is reduced. Marks constructed various kinds of 2-primal rings in [17] to give almost complete characterizations for 2-primal rings. Following Marks [16], a ring R is said to be *NI* if $N^*(R) = N(R)$. Note that R is NI if and only if $N(R)$ forms an ideal if and only if $R/N^*(R)$ is reduced. The NI condition is clearly a generalization of 2-primal rings. But NI rings need not be 2-primal by Birkenmeier et al. [5, Example 3.3], Hwang et al. [9, Example 1.2], or Marks [16, Example 2.2]. If a ring R is of bounded index of nilpotency, then R is NI if and only if R is 2-primal by [9, Proposition 1.4]. The upper triangular matrix rings over 2-primal (resp., NI) rings are basic examples of noncommutative 2-primal (resp., NI) rings which have important roles in noncommutative ring theory. Note that a ring R is reduced if and only if R is nil-semisimple (i.e., R has no nonzero nil ideals) and NI if and only if R is semiprime and 2-primal.

We consider next another generalization of 2-primal rings.

Definition 1.1. A ring R is said to be *quasi-2-primal* if $N_*(R) = N^*(R)$.

It is clear that a ring R is 2-primal if and only if it is both NI and quasi-2-primal. The following shows that quasi-2-primal rings need not be 2-primal and that NI property and quasi-2-primal property are independent of each other.

EXAMPLE 1.2. (1) Let S be any semiprimitive ring (i.e., $J(S) = 0$) and $R = Mat_n(S)$ for $n \geq 2$. Then $J(R) = Mat_n(J(S)) = 0$, entailing $J(R) = N_*(R) = N^*(R) = 0$. Thus R is quasi-2-primal. But R is not NI (hence not 2-primal) by the existence of $E_{ij} \in N(R)$, where $i \neq j$.

(2) We apply the construction and argument in [9, Example 1.2]. Let S be a 2-primal ring, n be a positive integer, and R_n be the 2^n by 2^n upper triangular matrix ring over S. Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, then R_n can be considered as a subring of R_{n+1} via σ. Notice that $D = \{R_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, \ldots\}$. Set $R = lim_{n\to\infty} R_n$ be the direct limit of D. Then $R = \cup_{i=1}^{\infty} R_n$, R is a semiprime ring by [10, Theorem 2.2(1)], and moreover R is NI by [9, Proposition 1.1 and Example 1.2]. Note $0 \neq N^*(R) = \{(a_{ij}) \in R \mid \ a_{ii} = 0 \text{ for all } i\} = N(R)$. So R is not quasi-2-primal because $N_*(R) = 0 \subsetneq N^*(R)$, entailing that R is not 2-primal.

We see next that an important ring theoretic property holds for the quasi-2-primal rings. The Köthe's conjecture (i.e., in any ring, the sum of two nil left ideals is nil) is an open problem in noncommutative ring theory that was raised by Gottfried Köthe in 1930 [13]. Various equivalent formulations are investigated by many authors, and one is that for any ring R and any nil ideal J of R, $Mat_n(J)$ is a nil ideal of $Mat_n(R)$ for every n (shown by Krempa [14] and Sands [21] independently). The Köthe's conjecture holds for NI rings by the definition.

It is well-known that $N_*(Mat_n(R)) = Mat_n(N_*(R))$ for any ring R.

PROPOSITION 1.3. *The Köthe's conjecture holds for quasi-2-primal rings.*

Proof. Let R be a quasi-2-primal ring and $n \geq 1$. Then

$$Mat_n(N^*(R)) = Mat_n(N_*(R)) = N_*(Mat_n(R)) = N^*(Mat_n(R))$$

by Theorem 2.2(2) to follow. So, for any nil ideal J of R, $Mat_n(J)$ is contained in $N^*(Mat_n(R))$, and hence $Mat_n(J)$ is a nil ideal of $Mat_n(R)$. Thus the Köthe's conjecture holds for R by [14] or [21]. $\square$

Considering the structure of the ring R in Example 1.2(2), one may ask whether the quasi-2-primal property is closed under direct limits. However we get a negative answer as follows. The ring R_n in Example

1.2(2) is 2-primal (hence quasi-2-primal) for any positive integer n. But the direct limit R is not quasi-2-primal. For, if R is quasi-2-primal then R is both NI and quasi-2-primal; hence R is 2-primal, a contradiction to R being not 2-primal. However we can get an affirmative answer when we use full matrix rings in place of upper triangular matrix rings in Example 1.2(2).

Example 1.4. Let S be a semiprimitive ring, n be a positive integer, and R_n be the 2^n by 2^n full matrix ring over S. We use the construction in Example 1.2(2). Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, then R_n can be considered as a subring of R_{n+1} via σ. Notice that $D = \{R_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, \ldots\}$. Set $R = lim_{n\to\infty} R_n$ be the direct limit of D.

Then $R = \cup_{i=1}^{\infty} R_n$, and R is a semiprimitive (hence quasi-2-primal) ring by the following computation. Assume on the contrary that R is not semiprimitive. Then there exists $0 \neq A \in R$ such that $1 - AB$ is a unit for all $B \in R$. Thus there exists $k \geq 1$ such that $A \in R_k$ and $1 - AC$ is a unit for all $C \in R_k$. This implies $J(R_k) \neq 0$, a contradiction to R_k being semiprimitive.

Therefore R is semiprimitive, and this shows that R is quasi-2-primal.

In the following we examine some examples which give meaning to the existence of quasi-2-primal rings.

Example 1.5. (1) The case of $N_*(R) = N^*(R) \subsetneqq J(R)$. Let D be a simple domain and $R = D[[x]]$. Then $N_*(R) = N^*(R) = 0 \subsetneqq xD[[x]] = J(R)$.

(2) The case of $N_*(R) \subsetneqq N^*(R) = J(R)$. We apply the argument in Example 1.2(2). Let S be a simple domain, n be a positive integer and $R_n = D_{2^n}(S)$. Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, then R_n can be considered as a subring of R_{n+1} via σ. Notice that $D = \{R_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, \ldots\}$. Set $R = lim_{n\to\infty} R_n$ be the direct limit of D. Then $R = \cup_{i=1}^{\infty} R_n$, and R is a prime ring by applying the method in the proof of [9, Proposition 1.3], entailing $N_*(R) = 0$. It is clear that $N^*(R) = \{A \in R \mid \text{ the diagonal entries of } A \text{ are zero}\}$. Moreover $N^*(R) = J(R)$ since $R/N^*(R) \cong \prod_{i=1}^{\infty} S_i$ where $S_i = S$ for all i.

(3) The case of $N_*(R) \subsetneqq N^*(R) \subsetneqq J(R)$. Let $S = T[[x]]$ for a simple domain T and R be the ring of the same structure as in (2). Then R is a prime ring by applying the method in the proof of [9, Proposition 1.3], entailing $N_*(R) = 0$. It is clear that $N^*(R) = \{A \in R \mid \text{ the diagonal entries of } A \text{ are zero}\}$. Moreover

$$J(R) = \{A \in R \mid \text{ the diagonal entries of } A \text{ are in xT[[x]]}\}$$

since $R/N^*(R) \cong T[[x]]$.

Following Neumann [18], a ring R is called *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$. Regular rings are semiprimitive by [8, Corollary 1.2(c)] and so quasi-2-primal. The classes of 2-primal rings and NI rings are closed under subrings by [4, Proposition 2.2] and [9, Proposition 2.4], respectively. But the class of quasi-2-primal rings is not closed under subrings as the following shows.

Example 1.6. We apply the argument in Example 1.2(2). Let S be a regular ring, n be a positive integer, and $R_n = Mat_{2^n}(S)$. Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$, then R_n can be considered as a subring of R_{n+1} via σ. Notice that $D = \{R_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, \ldots\}$. Set $R = lim_{n\to\infty} R_n$ be the direct limit of D. Then R is also a regular (hence quasi-2-primal) ring. For, letting $A \in R$, $A \in R_k$ for some $k \geq 1$, and so $A = ABA$ for some $B \in R_k \subset R$ since R_k is regular.

Next let $R'_n = U_{2^n}(S)$ and define a map $\sigma : R'_n \to R'_{n+1}$ by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. Let R' be the direct limit of $\{R'_n, \sigma_{nm}\}$. Then R' is not quasi-2-primal by Example 1.2(2), in spite of R' being a subring of R.

2. Structure of quasi-2-primal rings

In this section we study the structure of quasi-2-primal rings, and observe several sorts of ring extensions over quasi-2-primal rings which we meet usually in the study of ring theory.

We recall first the following which do roles in our study.

Lemma 2.1. (1) [15, Theorem 10.19] $N_*(R)[x] = N_*(R[x])$ *for any ring* R.

(2) [7, Corollary 5] $N_0(R)[x] = N_0(R[x])$ *for any ring* R.

As important cases, the quasi-2-primal property can pass to matrix rings and polynomial rings as the following shows.

Theorem 2.2. (1) *A ring R is quasi-2-primal then so is $R[x]$.*
(2) *A ring R is quasi-2-primal then so is $Mat_n(R)$.*
(3) *A ring R is quasi-2-primal then so is $U_n(R)$.*
(4) *A ring R is quasi-2-primal then so is $L_n(R)$.*
(5) *Let R, S be rings and ${}_RM_S$ an (R,S)-bimodule. Let $E = \begin{pmatrix} R & M \\ 0 & S \end{pmatrix}$. Then E is quasi-2-primal if and only if R and S are both quasi-2-primal.*

Proof. (1) It suffices to show the case of $X = \{x\}$. Let R be a quasi-2-primal ring. Then $N_*(R) = N^*(R)$. By Lemma 2.1(1), $N_*(R)[x] = N_*(R[x])$ and so $N_*(R[x]) = N_*(R)[x] = N^*(R)[x]$. Moreover $J(R[x]) = I[x]$ for some nil ideal I of R by [20, Reproof of Amitsur's Theorem (2.5.23) after Lemma 2.5.41]. Thus we now have

$$J(R[x]) \supseteq N^*(R[x]) \supseteq N_*(R[x]) = N_*(R)[x] = N^*(R)[x] \supseteq I[x] = J(R[x]),$$

entailing $N_*(R[x]) = N^*(R[x]) = J(R[x])$. Thus $R[x]$ is quasi-2-primal.

(2) Note that $N_*(Mat_n(R)) = Mat_n(N_*(R))$. Let R be a quasi-2-primal ring. Then

$$N_*(Mat_n(R)) = Mat_n(N_*(R)) = Mat_n(N^*(R)) \supseteq N^*(Mat_n(R)) \supseteq N_*(Mat_n(R)).$$

Thus $Mat_n(R)$ is quasi-2-primal.

(3) Use induction and set $S = Mat_{n-1}(R)$. Let R be a quasi-2-primal ring. Then we obtain that $U_n(R)$ is also quasi-2-primal. The proof of (4) is similar to that of (3).

(5) By observing $N^*(E) = \begin{pmatrix} N^*(R) & M \\ 0 & N^*(S) \end{pmatrix}$, we have that $N^*(E) = N_*(E)$ if and only if $N^*(R) = N_*(R)$ and $N^*(S) = N_*(S)$. □

Due to Armendariz [3, Lemma 1], Rege et al. [19] called a ring *Armendariz* if $a_ib_j = 0$ for all i, j whenever $(\sum_{i=0}^m a_ix^i)(\sum_{j=0}^n b_jx^j) = 0$. Reduced rings are Armendariz by [3, Lemma 1]. If R is an Armendariz ring then $N(R)$ is a subring of R by [2, Proposition 2.7 and Theorem 3.2]. A ring is usually called *Abelian* if every idempotent is central. Armendariz rings are Abelian by the proof of [1, Theorem 6] (or [12, Lemma 7]).

Lemma 2.3. (1) [11, Lemma 2.3(5)] *If R is an Armendariz ring then $N_0(R) = N_*(R) = N^*(R)$.*

(2) [2, Corollary 5.2] *If R is an Armendariz ring then* $N(R)[x] = N(R[x])$.

Armendariz rings are quasi-2-primal by Lemma 2.3(1), comparing this with the fact that Armendariz rings need not be NI (hence not 2-primal) by [2, Example 4.8]. There exist many quasi-2-primal rings but not Armendariz. Let S be a semiprimitive ring and $R = Mat_n(S)$ for $n \geq 2$. Then R is semiprimitive (hence quasi-2-primal) as above, but R is not Armendariz since R is non-Abelian.

Following the literature, a ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for all $a, b \in R$. The class of 2-primal (resp., NI) rings are directly finite by [4, Proposition 2.10] (resp., [9, Proposition 2.7]). Abelian rings are easily shown to be directly finite. So one may conjecture naturally that quasi-2-primal rings are also directly finite. However the following erases the possibility.

EXAMPLE 2.4. Let F be a field and $\mathbb{V}$ be an infinite dimensional vector space over F with a basis $\{v_1, v_2, \ldots\}$. Consider the endomorphism ring $R = End_F(\mathbb{V})$ and define $f, g \in R$ such that $fv_1 = 0, fv_j = v_{j-1}$ for $j = 2, 3, \ldots$ and $gv_i = v_{i+1}$ for $i = 1, 2, \ldots$. Then R is a regular ring such that $fg = 1$ but $gf \neq 1$. So R is quasi-2-primal which is not directly finite.

As another example, consider next $U_n(R)$ for $n \geq 2$. Then $U_n(R)$ is quasi-2-primal by Theorme 2.2(3). Take $a = (a_{ij})$ and $b = (b_{ij})$ in $U_n(R)$ such that $a_{ii} = f$, $b_{ii} = g$ for all i, and $a_{ij} = b_{ij} = 0$ for i, j with $i \neq j$. Then $ab = 1$ but $ba \neq 1$; hence $U_n(R)$ is not directly finite.

Recall that subrings of quasi-2-primal rings need not be quasi-2-primal by Example 1.6, but we get an affirmative answer for ideals as the following shows. It is well-known that $N_*(I) = N_*(R) \cap I$ and $N^*(I) = N^*(R) \cap I$ when I is an ideal of R.

LEMMA 2.5. *Let R be a ring and I be a proper ideal of R. If R is quasi-2-primal then so is I as a ring without identity.*

Proof. Recall first that $N_*(I) = N_*(R) \cap I$ and $N^*(I) = N^*(R) \cap I$ when I is an ideal of R. Suppose that R is a quasi-2-primal ring. Then we have

$$N^*(I) = N^*(R) \cap I = N_*(R) \cap I = N_*(I).$$

Therefore I is quasi-2-primal. $\square$

Lemma 2.6. *Let R be a ring and I be a proper ideal of R. If both R/I and I are quasi-2-primal rings, then so is R.*

Proof. Assume that R/I and I are quasi-2-primal rings, and let RaR be a nil ideal of R (i.e., $a \in N^*(R)$). Write $\bar{R} = R/I$ and $\bar{r} = r + I$ with $r \in R$. Then $\bar{R}\bar{a}\bar{R} \subseteq N_*(\bar{R})$, and so $\bar{a}$ is strongly nilpotent in $\bar{R}$.

Consider a sequence $(a_i)_{i\geq 0}$ such that $a_0 = a$, and $a_{i+1} = a_i r_i a_i \in a_i R a_i$ ($r_i \in R$) for any $i \geq 0$. Then there exists a positive integer k such that $a_k \in I$, because $\bar{R}$ is quasi-2-primal. Note that Ia_kI is a nil ideal of I (i.e., $a_k \in N^*(I)$) because every a_k is contained in $N^*(R)$. Next consider the following sequence:

$$\begin{aligned}
b_0 &= a_k;\\
b_1 &= a_{k+2} = a_{k+1}r_{k+1}a_{k+1} = a_k(r_k a_k r_{k+1} a_k r_k)a_k \in a_k I a_k = b_0 I b_0;\\
b_2 &= a_{k+4} = a_{k+3}r_{k+3}a_{k+3} = a_{k+2}(r_{k+2}a_{k+2}r_{k+3}a_{k+2}r_{k+2})a_{k+2} \in a_{k+2}Ia_{k+2}\\
&= b_1 I b_1;\\
b_i &= a_{k+2i}\\
&= a_{k+2(i-1)}(r_{k+2(i-1)}a_{k+2(i-1)}r_{k+2i}a_{k+2(i-1)}r_{k+2(i-1)})a_{k+2(i-1)}\\
&\qquad \in a_{k+2(i-1)}Ia_{k+2(i-1)} = b_{i-1}Ib_{i-1},
\end{aligned}$$

where $i \geq 1$. Since I is quasi-2-primal, there exists $l \geq 0$ such that $b_l = 0$. This implies $a_{k+2l} = 0$, and so a is strongly nilpotent in R. Thus we now have $N^*(R) = N_*(R)$, and therefore R is quasi-2-primal. □

From the preceding lemmas we obtain the following results.

Proposition 2.7. *Let R be a ring and I be an ideal of R that is a direct summand of R. Then the following conditions are equivalent:*

(1) *R is quasi-2-primal;*

(2) *Both I and R/I are quasi-2-primal rings.*

Proof. Since I is a direct summand of R, the ring R is isomorphic to $I \oplus R/I$. Thus, (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) are shown by Lemma 2.5 and Lemma 2.6, respectively. □

Lemma 2.8. *Let $\{R_i | i \in I\}$ be a set of quasi-2-primal rings. Then the direct sum of $\{R_i | i \in I\}$ is quasi-2-primal.*

Proof. Let $\{R_i | i \in I\}$ be a set of quasi-2-primal rings and R be the direct sum of R_i's. Suppose that RaR is a nonzero nil ideal of R where

$a = (a_i) \in R$. Then there exist indices $i_1, i_2, \ldots, i_k$ in I such that $a_{i_j} \neq 0$ for $j = 1, \ldots, k$, and $a_i = 0$ for all $i \notin \{i_1, i_2, \ldots, i_k\}$.

For any i_j, $R_{i_j} a_{i_j} R_{i_j}$ is a nil ideal of R_{i_j}. Thus a_{i_j} is a strongly nilpotent element in R_{i_j}, i.e., $a_{i_j} \in N_*(R_{i_j})$ because every R_i is a quasi-2-primal ring. This implies that a is also strongly nilpotent because $\{i_j \mid j = 1, \ldots, k\}$ is finite. Therefore R is quasi-2-primal. □

PROPOSITION 2.9. *For a ring R, the following conditions are equivalent:*

(1) *R is quasi-2-primal;*

(2) *Both eR and $(1-e)R$ are quasi-2-primal rings for some nonzero central idempotent e of R.*

Proof. (1) $\Rightarrow$ (2) is shown by Lemma 2.5. (2) $\Rightarrow$ (1) is done by Lemma 2.8. □

Acknowledgments.

The authors thank the referee for very careful reading of the manuscript and many valuable suggestions that improved the paper by much.

References

[1] D.D. Anderson and V. Camillo, *Armendariz rings and Gaussian rings*, Comm. Algebra **26** (1998), 2265–2272.

[2] R. Antoine, *Nilpotent elements and Armendariz rings*, J. Algebra **319** (2008), 3128–3140.

[3] E.P. Armendariz, *A note on extensions of Baer and P.P.-rings*, J. Austral. Math. Soc. **18** (1974), 470–473.

[4] G.F. Birkenmeier, H.E. Heatherly, and E.K. Lee, *Completely prime ideals and associated radicals*, Proc. Biennial Ohio State-Denison Conference 1992, edited by S. K. Jain and S. T. Rizvi, World Scientific, Singapore-New Jersey-London-Hong Kong (1993), 102–129.

[5] G.F. Birkenmeier, J.Y. Kim, and J.K. Park, *Regularity conditions and the simplicity of prime factor rings*, J. Pure Appl. Algebra **115** (1997), 213–230.

[6] V. Camillo and P.P. Nielsen, *McCoy rings and zero-divisors*, J. Pure Appl. Algebra **212** (2008), 599–615.

[7] V. Camillo, C.Y. Hong, N.K. Kim, Y. Lee, and P.P. Nielsen, *Nilpotent ideals in polynomial and power series rings*, Proc. Amer. Math. Soc. **138** (2010), 1607–1619.

[8] K.R. Goodearl, *Von Neumann Regular Rings*, Pitman, London, 1979.

[9] S.U. Hwang, Y.C, Jeon, and Y. Lee, *Structure and topological conditions of NI rings*, J. Algebra **302** (2006), 186–199.

[10] Y.C. Jeon, H.K. Kim, Y. Lee, and J.S. Yoon, *On weak Armendariz rings*, Bull. Korean Math. Soc. **46** (2009), 135–146.
[11] N.K. Kim, K.H. Kim, and Y. Lee, *Power series rings satisfying a zero divisor property*, Comm. Algebra **34** (2006), 2205–2218.
[12] N.K. Kim and Y. Lee, *Armendariz rings and reduced rings*, J. Algebra **223** (2000), 477–488.
[13] G. Köthe, *Die Struktur der Ringe deren Restklassenring nach dem Radikal vollstanding irreduzibel ist*, Math. Z. **32** (1930), 161–186.
[14] J. Krempa, *Logical connections among some open problems concerning nil rings*, Fund. Math. **76** (1972), 121–130.
[15] T.Y. Lam, *A First Course in Noncommutative Rings*, Springer-Verlag, New York, 1991.
[16] G. Marks, *On* 2*-primal Ore extensions*, Comm. Algebra **29** (2001), 2113–2123.
[17] G. Marks, *A taxonomy of* 2*-primal rings*, J. Algebra **266** (2003), 494–520.
[18] J.V. Neumann, *On regular rings*, Proceedngs of the National Academy of Sciences, **22** (1936), 707–713.
[19] M.B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (1997), 14–17.
[20] L.H. Rowen, *Ring Theory*, Academic Press, San Diego, 1991.
[21] A.D. Sands, *Radical and Morita contex*, J. Algebra **24** (1973), 335–345.

Hai-lan Jin
Department of Mathematics
Yanbian University
Yanji 133002, China
E-mail: hljin98@ybu.edu.cn

Zhelin Piao
Department of Mathematics
Yanbian University
Yanji 133002, China
E-mail: zlpiao@ybu.edu.cn

Sang Jo Yun
Department of Mathematics
Dong-A University
Busan, 49315, Korea
E-mail: sjyun@dau.ac.kr