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MAPS PRESERVING m- ISOMETRIES ON HILBERT SPACE

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Abstract. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. In this paper, we prove that if $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a unital surjective bounded linear map, which preserves m- isometries $m = 1, 2$ in both directions, then there are unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{tr}V$$

for all $T \in \mathcal{B}(\mathcal{H})$, where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

1. Introduction

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are linear spaces and $\varphi : X \to Y$ is a map. We say that φ is a preserving map in both directions whenever

$$x \in \mathcal{X} \text{ has the property } p \Leftrightarrow \varphi(x) \in \mathcal{Y} \text{ has the property } p.$$

Linear preserver problems concern the characterization of linear operators on matrix spaces that leave certain functions, subsets, relations, etc., invariant. These problems often study the form of linear maps preserving some properties in Banach algebras or other linear spaces. The earliest papers on linear preserver problems date back to 1897, and a great deal of effort has been devoted to the study of this type of question

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since then. Many mathematicians have investigated several linear preserver problems, see [6,8,9]. In [2,5] and [4] the authors study the linear preserving maps on Hilbert and Hilbert module space, respectively. Suppose that $\mathcal{H}$ is a complex Hilbert space. We assume that $\mathcal{X} = \mathcal{Y} = \mathcal{B}(\mathcal{H})$ (the set of all bounded linear operators on $\mathcal{H}$) and then, we expose a structure of surjective continuous linear maps $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, which preserve m-isometries $m = 1, 2$ in $\mathcal{B}(\mathcal{H})$ in both directions. "Note that we say the operator $T \in \mathcal{B}(\mathcal{H})$ is m-isometry $m = 1, 2$, if the operator $T \in \mathcal{B}(\mathcal{H})$ is both 1-isometry and 2-isometry".

Definition 1.1. [1] A bounded linear operator T on a complex Hilbert space $\mathcal{H}$ is called an m-isometry if it satisfies

$$\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} T^{*j} T^{j} = 0, \qquad m \geq 1.$$

Example 1.2. Let $T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and be $*$ be the conjugate transpose. We have

$$T^{*} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; T^{*2} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \; T^{2} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix},$$
$$T^{*3} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \; T^{3} = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

With a simple calculation, we have

$$T^{*3}T^{3} - 3T^{*2}T^{2} + 3T^{*}T - I = 0,$$

thus T is 3-isometry.

Recall that if $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is a Hilbert space and K is a nonzero positive invertible element of $\mathcal{B}(\mathcal{H})$ and we define $\langle x, y \rangle_K = \langle Kx, y \rangle$ for each $x, y \in \mathcal{H}$, then $\langle \cdot, \cdot \rangle_K$ turns into an inner product on $\mathcal{H}$ and $\mathcal{H}_K = (\mathcal{H}, \langle \cdot, \cdot \rangle_K)$ becomes Hilbert space, too.

If T^* is the adjoint of T with respect to the inner product $\langle \cdot, \cdot \rangle$, then $T^{\sharp} = K^{-1}T^*K$ is the K-adjoint of T with respect to the inner product $\langle \cdot, \cdot \rangle_K$. It is easy to see that $\sharp$ is an involution on $\mathcal{B}(\mathcal{H})$. We say that $S \in \mathcal{B}(\mathcal{H})$ is K-self-adjoint if $S^{\sharp} = S$. The set of all bounded linear operators on $\mathcal{H}$ with respect to inner product $\langle ., . \rangle_K$ is the same as $\mathcal{B}(\mathcal{H}, \langle ., . \rangle)$.

DEFINITION 1.3. A bounded linear operator T on a complex Hilbert space $\mathcal{H}$ is called a K-m-isometry if it satisfies

$$\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}T^{\sharp j}T^{j}=0, \qquad m\geq 1$$

and it is called K-unitary, if $TK^{-1}T^{*}K=K^{-1}T^{*}KT=I$.

DEFINITION 1.4. Let $\mathcal{A}$ be a C^{*}-algebra. Denote by $\mathcal{Z}(\mathcal{A})$ the centre of $\mathcal{A}$, namely $\mathcal{Z}(\mathcal{A})=\{a\in\mathcal{A}\mid ab=ba\ \forall\ b\in\mathcal{A}\}$.

Utilizing [3, p.47] and [7, p.158], one can easily conclude the following lemma.

LEMMA 1.5. *Suppose that $\mathcal{A}$ is a C^{*}-algebra. Then the following conditions are equivalent:*
(i) For all $a,b\in\mathcal{A}$, $a\mathcal{A}b=\{0\}$ implies $a=0$ or $b=0$.
(ii) For all ideals I and J of $\mathcal{A}$, $IJ=\{0\}$ implies $I=\{0\}$ or $J=\{0\}$.
(iii) For all closed ideals I and J of $\mathcal{A}$, $IJ=\{0\}$ implies $I=\{0\}$ or $J=\{0\}$.

Recall that a C^{*}-algebra is said to be prime if it satisfies one of the conditions of Lemma 1.5. In particular it shows that topological and algebraic primeness are equivalent in the setting of C^{*}-algebras.

LEMMA 1.6. [5] *The set $\mathcal{B}(\mathcal{H})$ is a prime C^{*}-algebra.*

A linear map φ from a C^{*}-algebra $\mathcal{A}$ into a C^{*}-algebra $\mathcal{B}$ is called a $*$-Jordan homomorphism if $\varphi(a^{2})=\varphi(a)^{2}$ and $\varphi(a^{*})=\varphi(a)^{*}$ for every $a\in\mathcal{A}$. A well known result of Herstein [3, Theorem 3.1] states that a $*$-Jordan homomorphism onto a prime C^{*}-algebra is either a $*$-homomorphism or a $*$-anti-homomorphism.

2. Linear maps that preserve m-isometries

In this section, we intend to characterize the unital surjective linear maps from $\mathcal{B}(\mathcal{H})$ onto itself that preserve m-isometries. We need the following known theorem.

THEOREM 2.1. [6, p.208] *Suppose that $\mathcal{H}$ is a Hilbert space. If $\varphi:\mathcal{B}(\mathcal{H})\to\mathcal{B}(\mathcal{H})$ is a surjective linear isometry, then there are unitary operators U and V in $\mathcal{B}(\mathcal{H})$ such that φ is either of the form*

$$\varphi(T)=UTV$$

or of the form

$$\varphi(T) = UT^{tr}V$$

for each $T \in \mathcal{B}(H)$, where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

LEMMA 2.2. [2] *Let $\mathcal{H}$ be a separable complex Hilbert space. If $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a continuous and surjective homomorphism or anti-homomorphism, then φ is an injection .*

To achieve our next result, we utilize the strategy of [4].

THEOREM 2.3. *Let $\mathcal{H}$ be a separable complex Hilbert space and let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves m-isometries $m = 1, 2$, in both directions, then there are unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{tr}V,$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Proof. Pick a self-adjoint operator S in $\mathcal{B}(\mathcal{H})$. Then $\exp(itS)^* = \exp(-itS)$ for $t \in \mathbb{R}$. Clearly the operator $\exp(itS)$ is m-isometry for $m = 1, 2$. Therefore

$$\begin{cases} -I + \varphi(\exp(itS))^*\varphi(\exp(itS)) = 0, \\ I - 2\varphi(\exp(itS))^*\varphi(\exp(itS)) + \varphi(\exp(itS))^{*2}\varphi(\exp(itS))^2 = 0. \end{cases}$$

Thus

$$-\varphi(\exp(itS))^*\varphi(\exp(itS)) + \varphi(\exp(itS))^{*2}\varphi(\exp(itS))^2 = 0,$$

hence

$$\begin{aligned} &\varphi(I + itS + \frac{(it)^2}{2!}S^2 + ...)^{*2}\varphi(I + itS + \frac{(it)^2}{2!}S^2 + ...)^2 \\ &= \varphi(I + itS + \frac{(it)^2}{2!}S^2 + ...)^*\varphi(I + itS + \frac{(it)^2}{2!}S^2 + ...). \end{aligned}$$

And

$$\begin{aligned} &(I - it\varphi(S)^* - \frac{t^2}{2}\varphi(S^2)^* + ...)^2(I + it\varphi(S) - \frac{t^2}{2}\varphi(S^2) + ...)^2 \\ &= (I - it\varphi(S)^* - \frac{t^2}{2}\varphi(S^2)^* + ...)(I + it\varphi(S) - \frac{t^2}{2}\varphi(S^2) + ...), \end{aligned}$$

so

$$I + 2it(\varphi(S) - \varphi(S)^*) + t^2(4\varphi(S)^*\varphi(S) - \varphi(S^2) - \varphi(S)^2 - \varphi(S^2)^* - \varphi(S)^{*2}) + ...$$
$$= I + it(\varphi(S) - \varphi(S)^*) + t^2(\varphi(S)^*\varphi(S) - \frac{1}{2}\varphi(S^2) - \frac{1}{2}\varphi(S^2)^*) + ...$$

Similar to proof of [4, Theorem 2.6.], we have
(i) $\varphi(T^*) = \varphi(T)^*$,
(ii) $\varphi(T^2) = \varphi(T)^2$,
for each $T \in \mathcal{B}(\mathcal{H})$. Therefore, φ is a $*$-Jordan homomorphism. It is known that every $*$-Jordan homomorphism onto a prime algebra is a $*$-homomorphism or a $*$-anti-homomorphism. Since $\mathcal{B}(\mathcal{H})$ is a prime algebra, φ is a $*$-homomorphism or a $*$-anti-homomorphism. Also by Lemma 2.2, since φ is injection, it is a $*$-automorphism or a $*$-anti-automorphism. By Theorem 2.1 the proof is complete. □

THEOREM 2.4. *Let $\mathcal{H}$ be a complex Hilbert space and let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves m-isometries $m = 1, 2$, in both directions, then there are unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that*

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{tr}V,$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Proof. At first, we prove that φ is injection. Let $S \in \mathcal{B}(\mathcal{H})$ be self-adjoint and $\varphi(S) = 0$. Then $\varphi(S + I) = I$ and $\varphi(S - I) = -I$, since $\varphi(I) = I$. Clearly I and $-I$, for $m = 1, 2$, are m-isometries and since φ preserves m-isometries ($m = 1, 2$) in both directions, $S + I$ and $S - I$ are m-isometries for $m = 1, 2$. So $2S^4 + 10S^2 = 0$ and $2S^3 + S = 0$. Thus $S = 0$. Let $T \in \mathcal{B}(\mathcal{H})$ be an arbitrary element and $\varphi(T) = 0$. There exist self-adjoint operators $S_1, S_2 \in \mathcal{B}(\mathcal{H})$ such that $T = S_1 + iS_2$ and

$$\varphi(S_1) + i\varphi(S_2) = \varphi(T) = 0 = \varphi(T)^* = \varphi(S_1) - i\varphi(S_2).$$

So $\varphi(S_1) = 0$ and $\varphi(S_2) = 0$, therefore $S_1 = S_2 = 0$, hence $T = 0$, which implies that φ is an injection. The rest proof is similar to Theorem 2.3. □

COROLLARY 2.5. *Let $\mathcal{H}$ be a complex Hilbert space and let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves*

K-m-isometries $m = 1, 2$, in both directions, then there are K-unitary operators $U, V \in \mathcal{B}(\mathcal{H})$ such that

$$\varphi(T) = UTV \quad \text{or} \quad \varphi(T) = UT^{tr}V,$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

In Theorems 2.3 and 2.4, if $V = U^{-1}$, then $\varphi(T) = UTU^{-1}$ or $\varphi(T) = UT^{tr}U^{-1}$. Since $\varphi(T^*) = \varphi(T)^*$,

$$\begin{aligned} UT^*U^{-1} &= (UTU^{-1})^* \\ &= (U^*)^{-1}T^*U^* \end{aligned}$$

A straightforward computation shows that

$$U^*UT^* = T^*U^*U$$

for each $T^* \in \mathcal{B}(\mathcal{H})$. Hence $U^*U \in \mathcal{Z}(\mathcal{B}(\mathcal{H}))$. We know that $\mathcal{Z}(\mathcal{B}(\mathcal{H})) = \{\lambda I : \lambda \in \mathbb{C}\}$. Therefore there is $\lambda \in \mathbb{C}$ such that $U^*U = \lambda I$. Moreover U is invertible, so $UU^* = \lambda I$. On the other hand the operator UU^* is self-adjoint. Then $\lambda = \pm 1$.

Corollary 2.6. *Let $\mathcal{H}$ be a Hilbert space and let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves m-isometries $m = 1, 2$, in both directions then there exist $\lambda = \pm 1$ and a unitary operator $U \in \mathcal{B}(\mathcal{H})$ satisfying $UU^* = U^*U = \lambda I$ such that*

$$\varphi(T) = \lambda UTU^{-1} \quad \text{or} \quad \varphi(T) = \lambda UT^{tr}U^{-1},$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

Also by the same way as mentioned above, we have

Corollary 2.7. *Let $\mathcal{H}$ be a Hilbert space and let $\varphi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a unital surjective bounded linear map. If φ preserves k-m-isometries $m = 1, 2$, in both directions then there exist $\lambda = \pm 1$ and a k-unitary operator $U \in \mathcal{B}(\mathcal{H})$ satisfying $UU^\sharp = U^\sharp U = \lambda I$ such that*

$$\varphi(T) = \lambda UTU^{-1} \quad \text{or} \quad \varphi(T) = \lambda UT^{tr}U^{-1},$$

where T^{tr} is the transpose of T with respect to an arbitrary but fixed orthonormal basis of $\mathcal{H}$.

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