Abstract. This article concerns a property of local rings and domains. A ring $R$ is called weakly local if for every $a \in R$, $a$ is regular or $1-a$ is regular, where a regular element means a non-zero-divisor. We study the structure of weakly local rings in relation to several kinds of factor rings and ring extensions that play roles in ring theory. We prove that the characteristic of a weakly local ring is either zero or a power of a prime number. It is also shown that the weakly local property can go up to polynomial (power series) rings and a kind of Abelian matrix rings.

Preliminary

Throughout this paper all rings are associative with identity unless otherwise stated. Let $R$ be a ring. We use $C(R)$ and $U(R)$ to denote the monoid of regular elements and the group of units in $R$, respectively. Let $J(R)$, $I(R)$, $N^*(R)$ and $N(R)$ denote the Jacobson radical, the set of all idempotents, the upper nilradical and the set of all nilpotent elements in $R$, respectively. $\mathbb{Z}(\mathbb{Z}_n)$ denotes the ring of integers (modulo $n$), and let $\mathbb{Q}$ be the field of rational numbers. Denote the $n$ by $n$ full (resp., upper triangular) matrix ring over $R$ by $\text{Mat}_n(R)$ (resp., $T_n(R)$). $I_n$ denotes the identity matrix in $\text{Mat}_n(R)$, and $E_{ij}$ denotes the matrix with $(i,j)$-entry 1 and elsewhere 0, and write $D_n(R) = \{(a_{ij}) \in T_n(R) \mid a_{11} = \cdots = a_{nn} =$
0}. For $S \subseteq R$, the right (resp., left) annihilator of $S$ is $R$ is denoted by $r_R(S)$ (resp., $l_R(S)$); that is $l_R(S) = \{ r \in R \mid rs = 0 \text{ for all } s \in S \}$ and $r_R(S) = \{ r \in R \mid sr = 0 \text{ for all } s \in S \}$. If $S = \{ a \}$ then we write $l_R(a)$ (resp., $r_R(a)$). An element $a \in R$ is said to be right (resp., left) regular in $R$ if $r_R(a) = 0$ (resp., $l_R(a) = 0$). An element is called regular if it is both right and left regular. The characteristic of $R$ is denoted by $ch(R)$.

Given a set $S$, the cardinality of $S$ is expressed by $|S|$.

A ring is usually called reduced if it has no nonzero nilpotent elements. Due to Feller [1], a ring is called right (resp. left) duo if every right (resp. left) ideal is an ideal; a ring is called duo if it is both right and left duo. A ring is called Abelian if every idempotent is central. It is easily shown that both one-side duo rings and reduced rings are Abelian. Following Hong et al. [3], a ring $R$ is called right (resp., left) $DR$ if $C(R)a \subseteq aC(R)$ (resp., $aC(R) \subseteq C(R)a$) for all $a \in R$; and a ring is called $DR$ if it is both left and right $DR$. So a ring $R$ is $DR$ if and only if $C(R)a = aC(R)$ for all $a \in R$. A ring $R$ is clearly $DR$ when $C(R) \subseteq Z(R)$. Right $DR$ rings are also Abelian by [3, Lemma 2.1(1)]. A ring $R$ is called local if $R/J(R)$ is a division ring. Every local ring $R$ is Abelian because for every $a \in R$, $a \in U(R)$ of $1 - a \in U(R)$.

1. Weakly local rings

We first deal with a ring property that is satisfied by domains and local rings. A ring $R$ shall be called weakly local if for every $a \in R$, $a \in C(R)$ or $1 - a \in C(R)$. The following does an important role throughout this article.

**Lemma 1.1.** (1) The class of weakly local rings contains domains and local rings.

(2) If $R$ is a weakly local ring then $I(R) = \{ 0, 1 \}$.

(3) Every weakly local ring is Abelian.

(4) The class of weakly local rings is closed under subring with the inherited identity.

(5) Commutative rings need not be weakly local.

**Proof.** (1) Domains are clearly weakly local. Let $R$ be a local ring and $a \in R$. Then $a \in U(R)$ or $1 - a \in U(R)$; hence $R$ is weakly local, since $U(R) \subseteq C(R)$.
(2) Let \( R \) be a weakly local ring and \( e \in I(R) \). Then \( e \in C(R) \) or \( 1 - e \in C(R) \); hence \( e = 0 \) or \( e = 1 \), noting that 1 is the only regular idempotent.

(3) is an immediate consequence of (2).

(4) Let \( R \) be a weakly local ring and \( S \) be a subring of \( R \) with the identity of \( R \). Suppose \( a \notin C(S) \). Then \( a \notin C(R) \). Since \( R \) is weakly local, \( 1 - a \in C(R) \) and \( 1 - a \in C(S) \) follows. Thus \( S \) is weakly local.

(5) Consider the ring \( \mathbb{Z}_6 \) and take \( 3 \in \mathbb{Z}_6 \). Then \( 3 \notin C(\mathbb{Z}_6) \) and \( 1 - 3 = 4 \notin C(\mathbb{Z}_6) \).

From Lemma 1.1, we obtain that \( I(R) = \{0, 1\} \) (hence \( R \) is Abelian) for a local ring \( R \), and that commutative rings need not be local. In the following example we see relations between weakly local rings and related ring properties.

**Example 1.2.** (1) We refer to [9, Theorem 1.3.5, Corollary 2.1.14, and Theorem 2.1.15]. Let \( K \) be a field of characteristic zero and \( A = K \langle x, y \rangle \) be the free algebra with noncommuting indeterminates \( x, y \) over \( K \). Let \( I \) be the ideal of \( A \) generated by \( yx - xy - 1 \), and set \( R = A/I \). Then \( R \) is a noncommutative domain and so \( R \) is weakly local by Lemma 1.1(1). But \( R \) is neither right nor left duo as can be seen by the one-sided ideals \( xR \) and \( Rx \) which cannot be two-sided. Moreover \( R \) is neither right nor left DR by [3, Lemma 2.1(3)].

(2) The weakly local domain \( R \) in (1) is clearly not local, noting \( J(R) = 0 \) by [9, Theorem 1.3.8].

(3) The weakly local ring \( D_n(R) \) over a weakly local ring \( R \) \((n \geq 2)\), in Theorem 2.2 to follow, is clearly not a domain.

(4) Let \(|I| \geq 2 \) and \( R_i \) be rings for every \( i \in I \). Then the direct product of \( R_i \)'s cannot be weakly local by Lemma 1.1(2).

Right duo rings need not be right DR by [3, Example 1.4(2)]. It is not proved yet that right DR rings are right duo. The weakly local property can be a condition under which right DR can be right duo.

**Proposition 1.3.** Let \( R \) be a weakly local ring. If \( R \) is right DR then \( R \) is right duo.

*Proof.\* We apply the proof of [3, Proposition 1.6]. Let \( r, a \in R \) and consider \( ra \). Since \( R \) is weakly local, \( r \in C(R) \) or \( 1 - r \in C(R) \). Suppose that \( R \) is right DR. If \( r \in C(R) \) then \( ra = ar_1 \in aR \) for some \( r_1 \in C(R) \).
If \(1 - r \in C(R)\) then \((1 - r)a = ar_2\) for some \(r_2 \in C(R)\). This yields \(ra = a - ar_2 = a(1 - r_2) \in aR\). Thus \(R\) is right duo.

In the following we see a more general situation of the proof of Lemma 1.1(5).

**Theorem 1.4.** (1) Let \(n = p^k\) such that \(p\) is a prime number and \(k \geq 1\). Then \(Z_n\) is (weakly) local.

(2) Let \(n = p_1^{m_1}p_2^{m_2} \cdots p_k^{m_k}\) such that \(k \geq 2\), \(p_i\) is a prime number for all \(i\), \(p_i \neq p_j\) if \(i \neq j\), and \(m_i \geq 1\) for all \(i\). Then \(Z_n\) is not weakly local.

(3) Let \(R\) be a ring of \(ch(R) = p_1^{m_1}p_2^{m_2} \cdots p_k^{m_k}\) such that \(k \geq 2\), \(p_i\) is a prime number for all \(i\), \(p_i \neq p_j\) if \(i \neq j\), and \(m_i \geq 1\) for all \(i\). Then \(R\) is not weakly local.

**Proof.** (1) is easily verified.

(2) \(Z_n\) is isomorphic to \(Z_{p_1^{m_1}} \times Z_{p_2^{m_2}} \times \cdots Z_{p_k^{m_k}}\). So \(Z_n\) contains an idempotent that is neither zero not the identity. Thus \(Z_n\) is not weakly local by Lemma 1.1(2).

(2) By hypothesis, \(R\) contains \(Z_n\) as a subring with the same identity, where \(n = ch(R)\). But \(Z_n\) is not weakly local by (2), therefore \(R\) is not weakly local by Lemma 1.1(4).

From Theorem 1.4, we obtain the following.

**Corollary 1.5.** (1) The class of weakly local rings is not closed under factor rings.

(2) Let \(R\) be a weakly local ring. Then \(ch(R)\) is either zero or a power of a prime number.

**Proof.** (1) First note that \(Z\) is weakly local by Lemma 1.1(1). Let \(n = p_1^{m_1}p_2^{m_2} \cdots p_k^{m_k}\) such that \(k \geq 2\), \(p_i\) is prime for all \(i\), \(p_i \neq p_j\) if \(i \neq j\), and \(m_i \geq 1\) for all \(i\). Consider the ideal \(nZ\) and set \(R = Z/nZ\). Then \(R\) is not weakly local by Theorem 1.4(2).

(2) Suppose \(ch(R) \neq 0\). Then the proof is done by Theorem 1.4.

As another example of Corollary 1.5(1), let \(R_0\) be the localization of \(Z\) at the prime ideal \(pZ\), where \(p\) is an odd prime; and next set \(R\) be the quaternions over \(R_0\). Then \(R\) is clearly a domain (hence weakly local) and \(J(R) = pR\). But \(R/J(R)\) is isomorphic to \(Mat_2(Z_p)\) by the argument in [2, Exercise 2A]. But \(Mat_2(Z_p)\) is not weakly local by Lemma 1.1(2). Thus \(R/J(R)\) is not weakly local.
Compare Corollary 1.5(2) with the fact that the characteristic of a domain is either zero or a prime number. The following elaborates upon Theorem 1.4.

**Remark 1.6.** For a given ring $R$, identify $s \in \mathbb{Z} / (\mathbb{Z}_n)$ with $s \cdot 1$ in $R$, where 1 is the identity of $R$.

(1) Let $R$ be the ring in Theorem 1.4(2), i.e., $R = \mathbb{Z}/n\mathbb{Z}$ with $n = p_1^{m_1}p_2^{m_2}\cdots p_k^{m_k}$ such that $k \geq 2$, $p_i$ is a prime number for all $i$, $p_i \neq p_j$ if $i \neq j$, and $m_i \geq 1$ for all $i$. Consider $a = p_i^{m_i}$ for $i \in \{1, 2, \ldots, k\}$. Since $R$ is finite, there exist $s, t \geq 1$ such that $a^s = a^{s+t}$. Then $a^{st} \in I(R)$ by the proof of [5, Proposition 16]. Note that $a^{st} = p_i^{m_is}$ is nonzero since $k \geq 2$, and moreover $a^{st} \neq 1$ since $a^{st}(p_1^{m_1}\cdots p_i^{-1}p_{i+1}^{m_{i+1}}\cdots p_k^{m_k}) = 0$, noting $p_1^{m_1}\cdots p_{i-1}^{m_{i-1}}p_{i+1}^{m_{i+1}}\cdots p_k^{m_k} \neq 0$.

(2) Let $R$ be a ring of $\text{ch}(R) = m_1m_2\cdots m_k$ with $k \geq 2$ and $m_i \geq 2$ for all $i$. Suppose $m_{j+1} = m_j + 1$ for some $1 \leq j \leq k-1$. Note $m_{j+1} \notin C(R)$. Consider $l = 1 - m_{j+1}$. Since $l = l + 0 = l + m_1m_2\cdots m_k$, we get
\[
l = 1 - (m_j + 1) = -m_j + m_1m_2\cdots m_k = m_j(m_1\cdots m_{j-1}m_{j+1}\cdots m_k - 1)
\]
and hence we have
\[
l(m_1\cdots m_{j-1}m_{j+1}\cdots m_k)
= [m_j(m_1\cdots m_{j-1}m_{j+1}\cdots m_k - 1)](m_1\cdots m_{j-1}m_{j+1}\cdots m_k)
= [m_j(m_1\cdots m_{j-1}m_{j+1}\cdots m_k)](m_1\cdots m_{j-1}m_{j+1}\cdots m_k - 1)
= n(m_1\cdots m_{j-1}m_{j+1}\cdots m_k - 1) = 0,
\]
where $n = m_1m_2\cdots m_k$. So $1 - m_{j+1} \notin C(R)$ since noting $m_1\cdots m_{j-1}m_{j+1}\cdots m_k \neq 0$, and therefore $R$ is not weakly local.

Recall that a ring $R$ is called *semilocal* if $R/J(R)$ is semisimple Artinian, and that a ring $R$ is called *semiperfect* if $R$ is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are semiperfect by [8, Corollary 3.7.1].

**Proposition 1.7.** A ring $R$ is weakly local and semiperfect if and only if $R$ is local.

**Proof.** Let $R$ be weakly local and semiperfect. Since $R$ is semiperfect, $R$ has a finite orthogonal set $\{e_1, e_2, \ldots, e_n\}$ of local idempotents whose sum is 1 by [8, Proposition 3.7.2], say $R = \prod_{i=1}^n e_iR$ such that each $e_iRe_i$ is a local ring. But, Lemma 1.1(2), we must get $\{e_1, \ldots, e_n\} = \{1\}$ since
R is weakly local, so that R is local. The converse is shown by [8, Corollary 3.7.1].

It is easily checked that for a local ring R, J(R) contains N(R). So one may ask whether for a weakly local ring R, J(R) contains N(R). But the answer is negative by the following.

**Example 1.8.** We deal with a subring of Mat₂(Z₂)[x] for n ≥ 2. Set

\[ R = \mathbb{Z}_2 + x \text{Mat}_n(\mathbb{Z}_2)[x]. \]

Let \( f(x) \in R \). Note that \( 1 + g(x) \in C(R) \) for all \( g(x) \in x \text{Mat}_n(\mathbb{Z}_2)[x] \) by the argument in the proof of Theorem 2.3 to follow. So \( f(x) \in C(R) \) or \( 1 - f(x) \in C(R) \). Thus \( R \) is weakly local.

Next we claim \( J(R) = 0 \). Letting \( f_1(x) = 1 + g_1(x) \) with \( g_1(x) \in x \text{Mat}_n(\mathbb{Z}_2)[x], 1 - f_1(x) = -g_1(x) \notin U(R) \), so that \( f_1(x) \notin J(R) \). This yields \( J(R) \subseteq x \text{Mat}_n(\mathbb{Z}_2)[x] \). Let \( h(x) = \sum_{i=1}^{m} a_i x^i \in x \text{Mat}_n(\mathbb{Z}_2)[x] \) with \( a_m \neq 0 \). Since \( \text{Mat}_n(\mathbb{Z}_2) \) is simple, we have \( Rh(x)R \) contains

\[ \{ b_1 x + \cdots + b_{m+1} x^{m+1} + x^{m+2} \mid b_i \in \text{Mat}_n(\mathbb{Z}_2) \} \]

because \( R_0 a_m R_0 = \text{Mat}_n(\mathbb{Z}_2)x^2, \) where \( R_0 = \text{Mat}_n(\mathbb{Z}_2)x \). But

\[ 1 - (b_1 x + \cdots + b_{m+1} x^{m+1} + x^{m+2}) \notin U(R), \]

entailing that \( b_1 x + \cdots + b_{m+1} x^{m+1} + x^{m+2} \) is not contained in \( J(R) \). Thus result implies \( h(x) \notin J(R) \), concluding \( J(R) = 0 \). However \( N(R) \neq 0 \) as can be seen by the nilpotent polynomial \( E_{12}x \) in \( R \).

**2. Extensions of weakly local rings**

In this section we study the weakly local property of some kinds of ring extensions that play important roles in ring theory.

**Lemma 2.1.** [6, Lemma 2.1] Let \( R \) be a ring and \( 0 \neq a \in R \). Then \( a \) is right (resp., left) regular if and only if \( (a_{ij}) \in D_n(R) \) is right (resp., left) regular, where \( a_{ii} = a \).

We obtain the following useful information about the weakly local property.

**Theorem 2.2.** Let \( n \geq 2 \). A ring \( R \) is weakly local if and only if \( D_n(R) \) is weakly local.
Proof. Let $R$ be a weakly local ring and $A = (a_{ij}) \in D_n(R)$. Assume $(a_{ij}) \notin C(D_n(R))$. Then $a_{ii} \notin C(R)$ by Lemma 2.1. Since $R$ is weakly local, $1-a_{ii} \in C(R)$. This yields $I_n - A = (b_{ij})$ such that $b_{ii} = 1-a_{ii}$ and $b_{ij} = -a_{ij}$ for all $i, j$ with $i < j$. Then $I_n - A$ is regular by Lemma 2.1. So $D_n(R)$ is weakly local. The converse is proved by Lemma 1.1(4). \hfill \Box

Note that $I(D_n(R)) = \{0, I_n\}$ over the weakly local ring $R$ by Lemma 1.1(2) and [4, Lemma 2]. However neither $Mat_n(R)$ nor $T_n(R)$ cannot be weakly local over any ring $R$ by Lemma 1.1(3) for $n \geq 2$.

**Theorem 2.3.** For a ring $R$ the following conditions are equivalent:

1. $R$ is weakly local;
2. $R[[x]]$ is weakly local;
3. $R[x]$ is weakly local.

Proof. Let $f(x) = \sum_{i=s}^\infty a_i x^i \in R[[x]]$ with $s \geq 0$. If $a_s \in C(R)$ then $f(x) \in C(R[[x]])$. For, letting $f(x)g(x) = 0$ for $g(x) = \sum_{j=0}^\infty b_j x^j \in R[[x]]$, we first get $a_s b_0 = 0$ since $a_s \in C(R)$ and $b_0 = 0$ follows; and inductively we have $b_j = 0$ for all $j \geq 0$. Letting $g(x)f(x) = 0$, we get $g(x) = 0$ similarly. Thus

$$\{\sum_{i=s}^\infty a_i x^i \in R[[x]] \mid a_s \in C(R)\} \subseteq C(R[[x]]),$$

where $s \geq 0$. A similar argument yields

$$\{\sum_{i=s}^m a_i x^i \in R[x] \mid a_s \in C(R) \text{ or } a_m \in C(R)\} \subseteq C(R[x]),$$

where $0 \leq s \leq m$.

(1) $\Rightarrow$ (2). Let $R$ be weakly local and $f(x) = \sum_{i=s}^\infty a_i x^i \in R[[x]]$ with $a_s \neq 0$. Assume $f(x) = \sum_{i=s}^\infty a_i x^i \notin C(R[[x]])$. Then $a_s \notin C(R)$ by the above argument. Suppose $s = 0$. Since $R$ is weakly local, $1-a_s \in C(R)$ and so $1-f(x) = (1-a_0) - \sum_{i=1}^\infty a_i x^i \in C(R[[x]])$ by the above argument. Suppose $s \geq 1$. Then $1-f(x) = 1 - \sum_{i=s}^\infty a_i x^i \in U(R[[x]])$. Thus $R[[x]]$ is weakly local.

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are proved by Lemma 1.1(4). \hfill \Box

In Theorem 2.3, $I(R[[x]]) = I(R[x]) = I(R) = \{0, 1\}$ by Lemma 1.1(2) and [7, Lemma 8], implying that $R[[x]]$ and $R[x]$ are also Abelian. Recall that domains and local rings are both weakly local. Domains
clearly pass to polynomial rings, but this is not valid for local rings. Letting $D$ be a division ring, $D[x]$ is not local (since $J(D[x]) = 0$).

Let $R$ be a ring. $R[x; x^{-1}]$ denotes the Laurent polynomial ring in $x$ over $R$, i.e., every element of $R[x; x^{-1}]$ has a unique representation in the form $\sum_{i \in \mathbb{Z}} a_i x^i$ with all but finitely many coefficients being zero. $R[[x; x^{-1}]]$ denotes the Laurent power series ring in $x$ over $R$, i.e., every element of $R[[x; x^{-1}]]$ has a unique representation in the form $\sum_{i \in \mathbb{Z}} a_i x^i$ with $a_{-n} = 0$ for all but finitely many $n \geq 1$.

**Remark 2.4.** Let $R$ be a ring. Applying the argument in the proof of Theorem 2.3, we obtain

$$\left\{ \sum_{i=s}^{\infty} a_i x^i \in R[[x; x^{-1}]] \mid a_s \in C(R) \right\} \subseteq C(R[[x; x^{-1}]])$$

and

$$\left\{ \sum_{i=t}^{m} a_i x^i \in R[x; x^{-1}] \mid a_t \in C(R) \text{ or } a_m \in C(R) \right\} \subseteq C(R[x; x^{-1}]),$$

where $s \in \mathbb{Z}$ and $t \leq m \in \mathbb{Z}$.

Let $R$ be weakly local and suppose that $f(x) = \sum_{i=s}^{\infty} a_i x^i \notin C(R[[x; x^{-1}]]).$ Then $a_s \notin C(R)$ by the preceding argument. Since $R$ is weakly local, $1 - a_s \in C(R)$. So $x^s - f(x) = (1 - a_s)x^s + \sum_{i=s+1}^{\infty} (-a_i)x^i \in C(R[[x; x^{-1}]]).$ Next suppose that $g(x) = \sum_{i=t}^{m} a_i x^i \notin C(R[x; x^{-1}]).$ Then $a_t, a_m \notin C(R)$ by the above argument. Since $R$ is weakly local, $1 - a_t, 1 - a_m \in C(R).$ So $x^t - g(x) = (1 - a_t)x^t + \sum_{i=t+1}^{m} (-a_i)x^i, x^m - g(x) = \sum_{i=t}^{m-1} (-a_i)x^i + (1 - a_m)x^m \in C(R[x; x^{-1}]).$

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