POSTPROCESSING FOR THE RAVIART–THOMAS
MIXED FINITE ELEMENT APPROXIMATION OF THE
EIGENVALUE PROBLEM

KWANG-YEON KIM

ABSTRACT. In this paper we present a postprocessing scheme for
the Raviart–Thomas mixed finite element approximation of the sec-
ond order elliptic eigenvalue problem. This scheme is carried out
by solving a primal source problem on a higher order space, and
thereby can improve the convergence rate of the eigenfunction and
eigenvalue approximations. It is also used to compute a posteriori
error estimates which are asymptotically exact for the $L^2$ errors of
the eigenfunctions. Some numerical results are provided to confirm
the theoretical results.

1. Introduction

Given a bounded polygonal domain $\Omega \subset \mathbb{R}^2$, we consider the following
second order elliptic eigenvalue problem: find $(p, \lambda) \in H^1(\Omega) \times \mathbb{R}$ such
that $\int_\Omega \rho p^2 \, dx = 1$ and

$$
\begin{cases}
- \text{div}(A \nabla p) + cp = \lambda \rho p & \text{in } \Omega, \\
p = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(1)
Here $A \in W^{1,\infty}(\Omega)^{2 \times 2}$ is a symmetric and uniformly positive definite matrix, and the functions $c, \rho \in W^{1,\infty}(\Omega)$ satisfy $c \geq 0$ and $\rho \geq \rho_0 > 0$ for some constant $\rho_0$. The Dirichlet boundary condition is assumed for simplicity only, and subsequent results are easily extended to other boundary conditions.

Finite element methods for the eigenvalue problem (1) have been extensively investigated in literature; we refer to the excellent survey papers [2, 3] for a general discussion. The mixed finite element method of (1) involving the vector function $u = A \nabla p$ as well as $p$ is useful when computing vibration modes of a fluid in a displacement formulation and has been studied, for example, in [1, 4, 5, 8, 14]. In particular, in [7, 9, 13], the supercloseness property of the mixed finite element method proved in [12] for the elliptic source problem has been extended to the elliptic eigenvalue problem (1).

One of the most effective ways to improve the accuracy of finite element approximations of the eigenfunctions and eigenvalues is to further solve an extra auxiliary source problem on a finer mesh [17] or on a higher order space [16]. Chen et al. [7] proposed such a postprocessing method to improve the convergence rate of the lowest order Raviart–Thomas mixed finite element approximation. In [7] the auxiliary source problem is solved by the mixed finite element method, and so requires a superconvergent eigenfunction approximation $\Pi_h p_h$.

In this paper we present a postprocessing method which can improve the convergence rate of the Raviart–Thomas mixed finite element method of any order for the eigenvalue problem (1). The idea is the same as that of [7], but we solve the auxiliary source problem using a higher order conforming finite element for the primal formulation. In doing so, our method does not need any superconvergent eigenfunction approximation like $\Pi_h p_h$. Moreover, we can obtain a posteriori error estimates for the $L^2$ errors of the eigenfunctions which are asymptotically exact when the eigenfunctions are smooth enough. We remark that residual-based error estimators were derived in [1, 8, 11] for the lowest order Raviart–Thomas element but they possess no such property, although easier to compute.

The rest of the paper is organized as follows. In Section 2 we introduce the Raviart–Thomas mixed finite element approximation of the eigenvalue problem and recall some previous results. In Section 3 we present
our postprocessing method which is used to improve the eigenvalue approximation and to construct a posteriori error estimates. Finally, in Section 4 we report some numerical results which confirm the theoretical results.

2. Mixed Finite Element Method and Supercloseness Property

By introducing the vector function \( u = A \nabla p \), we can rewrite the eigenvalue problem (1) in the following mixed form

\[
\begin{aligned}
A^{-1} u - \nabla p &= 0 \quad \text{in } \Omega, \\
- \text{div } u + cp &= \lambda \rho p \quad \text{in } \Omega, \\
p &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]

Define the function space

\[
H(\text{div}, \Omega) = \{ v \in L^2(\Omega)^2 : \text{div } v \in L^2(\Omega) \}
\]
equipped with the norm

\[
\| v \|_{H(\text{div}, \Omega)}^2 = \| v \|_{0, \Omega}^2 + \| \text{div } v \|_{0, \Omega}^2.
\]

Then the weak formulation for the mixed form (2) reads as follows: find \((u, p, \lambda) \in H(\text{div}, \Omega) \times L^2(\Omega) \times \mathbb{R}\) such that \(\int_\Omega \rho p^2 \, dx = 1\) and

\[
\begin{aligned}
\int_\Omega A^{-1} u \cdot v \, dx + \int_\Omega \text{div } v \, p \, dx &= 0 \quad \forall v \in H(\text{div}, \Omega), \\
- \int_\Omega \text{div } u \, q \, dx + \int_\Omega cpq \, dx &= \lambda \int_\Omega \rho p \, q \, dx \quad \forall q \in L^2(\Omega).
\end{aligned}
\]

From [2] we know that the eigenvalue problem (3) has an increasing sequence of positive eigenvalues \(\{\lambda_m\}_{m=1}^\infty\) and the associated eigenfunctions \(\{(u_m, p_m)\}_{m=1}^\infty\) such that \(\lim_{m \to \infty} \lambda_m = \infty\) and \(\int_\Omega p_m p_j \, dx = \delta_{ij}\).

The finite element discretization of (3) is constructed by first introducing a shape-regular family of triangulations \(\{T_h\}_{h>0}\) such that \(\Omega = \bigcup_{K \in T_h} K\) for each \(h > 0\) and \(h = \max_{K \in T_h} h_K\), where \(h_K\) denotes the diameter of \(K\). For an integer \(k \geq 0\), the \(k\)-th order Raviart–Thomas mixed element over the triangulation \(T_h\) is defined by (see [6])

\[
V_h = \{ v \in H(\text{div}, \Omega) : v|_K \in RT_k(K) \quad \forall K \in T_h \},
\]

\[
W_h = \{ q \in L^2(\Omega) : q|_K \in P_k(K) \quad \forall K \in T_h \},
\]
where $P_k(K)$ is the space of all polynomials on $K$ whose total degree is not greater than $k$, and

$$RT_h(K) = (P_k(K))^2 + (x_1, x_2)P_h(K).$$

Now we define the Raviart–Thomas mixed finite element method of the eigenvalue problem (3): find $(u_h, p_h, \lambda_h) \in V_h \times W_h \times \mathbb{R}$ such that

$$f \int_{\Omega} \rho p_h^2 \, dx = 1$$

and

$$\begin{cases}
\int_{\Omega} A^{-1} u_h \cdot v_h \, dx + \int_{\Omega} \text{div} \, v_h \rho_p \, dx = 0 & \forall v_h \in V_h, \\
- \int_{\Omega} \text{div} \, u_h \, q_h \, dx + \int_{\Omega} \text{cp} q_h \, dx = \lambda_h \int_{\Omega} \rho_p q_h \, dx & \forall q_h \in W_h.
\end{cases}$$

The abstract theory of [2, 4, 5, 14] shows that if the eigenfunctions $(u, p)$ of (3) belong to $H^t(\Omega)^2 \times H^{t+1}(\Omega)$ for $t > 0$, then the following a priori error estimates hold with $s = \min\{k + 1, t\}$

$$\|u - u_h\|_{H^{(\text{div}, \Omega)}} = \|p - p_h\|_{0, \Omega} \leq Ch^s, \quad \|\lambda - \lambda_h\| \leq Ch^{2s}$$

for sufficiently small $h$. Hereafter $C$ represents a generic positive constant which do not depend on the mesh size $h$.

Finally, we state the supercloseness result from [7, 9, 13] between the eigenfunction approximation $p_h$ in (4) and the $L^2$-projection $P_h p$ of the exact eigenfunction $p$ in $W_h$. To this end, following [13], we assume that the elliptic source problem

$$\begin{cases}
- \text{div}(A \nabla w) + cw = g & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega
\end{cases}$$

satisfies the following regularity estimate for some $\gamma \in (0, 1]$

$$\|w\|_{1+\gamma, \Omega} \leq C\|g\|_{0, \Omega}.$$ 

According to the results of [10], we have $\gamma = 1$ if $\Omega$ is convex and $\gamma = \frac{\pi}{\omega} - \epsilon$ for any $\epsilon > 0$ if $\Omega$ is a nonconvex polygon whose maximum interior angle is $\omega < 2\pi$.

**Theorem 1.** (See [13, Corollary 3.3]) Assume that the eigenfunctions $(u, p)$ of (3) belong to $H^t(\Omega)^2 \times H^{t+1}(\Omega)$ and the regularity estimate (6) holds (with $\gamma \leq t$). Then the following supercloseness result holds with $s = \min\{k + 1, t\}$

$$\|P_h p - p_h\|_{0, \Omega} \leq C h^{s+\gamma}$$

for sufficiently small $h$. 
Remark 1. By virtue of Theorem 1, the a priori error estimate \( \| p - p_h \|_{0,\Omega} = O(h^s) \) given in (5) may be improved to
\[
\| p - p_h \|_{0,\Omega} \leq \| p - P_h p \|_{0,\Omega} + \| P_h p - p_h \|_{0,\Omega}
\leq C h^{\min\{k+1,t+1\}} + C h^{\min\{k+1,t+\gamma\}}.
\]
(7)
This was observed in Remark 4.2 of [13].

Based on the supercloseness result of Theorem 1, the authors of [7] were able to construct a superconvergent eigenfunction approximation \( \Pi_h p_h \) which is continuous piecewise linear and satisfies
\[
\| p - \Pi_h p_h \|_0 \leq C h^{2\gamma}
\]
for the lowest order case \( k = 0 \) and \( s = t = \gamma \). Then, in order to improve the convergence rate of the eigenpair approximation \( (u_h, p_h, \lambda_h) \), they proposed a postprocessing method which solves the auxiliary source problem obtained from (4) by replacing \( p_h \) by \( \Pi_h p_h \) in the right-hand side of the second equation, i.e., \( \lambda_h \int_\Omega \rho p_h q_h \, dx \to \lambda_h \int_\Omega \rho (\Pi_h p_h) q_h \, dx \). The auxiliary source problem thus obtained is solved by the mixed finite element method on a finer mesh [17] or on a higher order space [16]. In doing so, construction of a superconvergent eigenfunction approximation \( \Pi_h p_h \) is essential.

3. Postprocessing for Improving the Eigenpair Approximation

In this section we propose a postprocessing method to improve the eigenpair approximation \( (p_h, \lambda_h) \) computed by the mixed finite element method (4) for any order \( k \geq 0 \).

Postprocessing method: Let \( \Psi_h \subset H^1_0(\Omega) \) be the standard \( P_{k+2} \) conforming finite element space on \( T_h \) and define \( \psi_h \in \Psi_h \) to be the solution of the auxiliary source problem
\[
a(\psi_h,\phi_h) = \lambda_h b(p_h,\phi_h) \quad \forall \phi_h \in \Psi_h,
\]
where
\[
a(\psi,\phi) = \int_\Omega (A \nabla \psi \cdot \nabla \phi + c \psi \phi) \, dx, \quad b(\psi,\phi) = \int_\Omega \rho \psi \phi \, dx.
\]
It is important to note that the auxiliary source problem is solved using a higher order conforming finite element for the primal formulation, whereas a higher-order mixed finite element method is used in [7]. Clearly, our method is much more economical. Moreover, our method does not demand any additional construction of a superconvergent eigenfunction approximation like $\Pi_h p_h$. Finally, as will be shown below, our method not only improves the convergence rate of the eigenpair approximation but also yields asymptotically exact a posteriori error estimates for both $\|u - u_h\|_{0, \Omega}$ and $\|p - p_h\|_{0, \Omega}$ if the eigenfunctions are smooth enough. All of these assertions are based on the following error estimates for $\psi_h$.

**Theorem 2.** Let $\psi_h \in \Psi_h$ be defined by (8) and assume that the regularity estimate (6) holds. Then we have for any $\chi_h \in \Psi_h$

$$\|p - \psi_h\|_{1, \Omega} \leq C(\|p - \chi_h\|_{1, \Omega} + h\|p - P_h p\|_{0, \Omega} + \|P_h p - p_h\|_{0, \Omega} + |\lambda - \lambda_h|)$$
and

$$\|p - \psi_h\|_{0, \Omega} \leq C(h^2 \|p - \chi_h\|_{1, \Omega} + h\|p - P_h p\|_{0, \Omega} + \|P_h p - p_h\|_{0, \Omega} + |\lambda - \lambda_h|).$$

**Proof.** The standard variational formulation of the eigenvalue problem (1) and the equation (8) together leads to

$$a(p - \psi_h, \phi_h) = \int_\Omega (\lambda p - \lambda_h p_h) \rho \phi_h \, dx \quad \forall \phi_h \in \Psi_h,$$

and thus

$$a(\psi_h - \chi_h, \phi_h) = a(p - \chi_h, \phi_h) - \int_\Omega (\lambda p - \lambda_h p_h) \rho \phi_h \, dx.$$

The second term of the right-hand side can be written as

$$\int_\Omega (\lambda p - \lambda_h p_h) \rho \phi_h \, dx$$

$$= \int_\Omega \lambda(p - P_h p) \rho \phi_h \, dx + \int_\Omega (\lambda - \lambda_h) P_h p (\rho \phi_h) \, dx + \int_\Omega \lambda_h (P_h p - p_h) \rho \phi_h \, dx$$

$$= \int_\Omega \lambda(p - P_h p) \{\rho \phi_h - P_h (\rho \phi_h)\} \, dx$$
$$+ \int_\Omega (\lambda - \lambda_h) P_h p (\rho \phi_h) \, dx + \int_\Omega \lambda_h (P_h p - p_h) \rho \phi_h \, dx.$$
Now applying Hölder’s inequality and the estimate \( \| v - P_h v \|_{0, \Omega} \leq Ch |v|_{1, \Omega} \), we obtain

\[
(13) \quad \left| \int_\Omega (\lambda p - \lambda_h p_h) \phi_h \, dx \right| \leq C(h \| p - P_h p \|_{0, \Omega} + |\lambda - \lambda_h| + \| P_h p - p_h \|_{0, \Omega}) \| \phi_h \|_{1, \Omega}.
\]

Then, by taking \( \phi_h = \psi_h - \chi_h \) in (12), it follows that

\[
\| \psi_h - \chi_h \|_{1, \Omega} \leq C(\| p - \chi_h \|_{1, \Omega} + h \| p - P_h p \|_{0, \Omega} + |\lambda - \lambda_h| + \| P_h p - p_h \|_{0, \Omega}),
\]

which gives the first result (9) by the triangle inequality.

To derive (10), we use the well-known duality argument. Let \( \xi \in H^1_0(\Omega) \) be the solution of

\[
a(\phi, \xi) = \int_\Omega (p - \psi_h) \phi \, dx \quad \forall \phi \in H^1_0(\Omega)
\]

such that \( \| \xi \|_{1+\gamma, \Omega} \leq C \| p - \psi_h \|_{0, \Omega} \) (see (6)). Then we have

\[
\| p - \psi_h \|_{0, \Omega}^2 = a(p - \psi_h, \xi) = a(p - \psi_h, \xi - I^1_h \xi) + a(p - \psi_h, I^1_h \xi),
\]

where \( I^1_h \xi \in H^1_0(\Omega) \) denotes the standard linear interpolant of \( \xi \). The first term of the right-hand side is easily bounded using the approximation property of \( I^1_h \)

\[
a(p - \psi_h, \xi - I^1_h \xi) \leq C \| p - \psi_h \|_{1, \Omega} \| \xi - I^1_h \xi \|_{1, \Omega} \leq C h^{\gamma} \| p - \psi_h \|_{1, \Omega} \| \xi \|_{1+\gamma, \Omega}.
\]

The second term is identical to (11) with \( \phi_h = I^1_h \xi \). Hence it follows by (13) that

\[
a(p - \psi_h, I^1_h \xi) \leq C(h \| p - P_h p \|_{0, \Omega} + |\lambda - \lambda_h| + \| P_h p - p_h \|_{0, \Omega}) \| I^1_h \xi \|_{1, \Omega}
\]

\[
\leq C(h \| p - P_h p \|_{0, \Omega} + |\lambda - \lambda_h| + \| P_h p - p_h \|_{0, \Omega}) \| \xi \|_{1+\gamma, \Omega}.
\]

The proof is completed by combining the above results.

As a corollary of Theorem 2, we obtain the following estimates which show that \( \psi_h \) indeed improves the convergence rate of the eigenfunction approximation \( p_h \) (if the eigenfunctions are smooth enough).

**Corollary 1.** Under the conditions of Theorem 1, we have for sufficiently small \( h \)

\[
\| p - \psi_h \|_{1, \Omega} \leq C h^{\min\{k+1+\gamma, t\}}, \quad \| p - \psi_h \|_{0, \Omega} \leq C h^{\min\{k+1+\gamma, t+\gamma\}}.
\]
Proof. Using the standard error estimates for \( \| p - \chi_h \|_{1, \Omega} \) and \( \| p - \mathcal{P}_h p \|_{0, \Omega} \), the error estimate (5) for \( |\lambda - \lambda_h| \) and Theorem 1 for \( \| \mathcal{P}_h p - \mathcal{P}_h p \|_{0, \Omega} \), we obtain from (9)
\[
\| p - \psi_h \|_{1, \Omega} \leq C h^{\min\{k+1, t\}} + C h^{1+\min\{k+1, t+1\}} + C h^{\min\{k+1, t\}+\gamma} + C h^{2\min\{k+1, t\}},
\]
which proves the first result. The second result follows similarly from (10).

**Eigenvalue improvement:** The superconvergent eigenfunction \( \psi_h \) can be used to improve the eigenvalue approximation via the Rayleigh quotient (see, e.g., [15])
\[
\hat{\lambda}_h = \frac{a(\psi_h, \psi_h)}{b(\psi_h, \psi_h)}.
\]
Estimation of the improved eigenvalue error \( \lambda - \hat{\lambda}_h \) depends on the following identity from [2]
\[
\frac{a(\phi, \phi)}{b(\phi, \phi)} - \lambda = \frac{a(p - \phi, p - \phi)}{b(\phi, \phi)} - \lambda \frac{b(p - \phi, p - \phi)}{b(\phi, \phi)},
\]
where \((p, \lambda)\) is an eigenpair of (1) and \( \phi \) is any nonzero function in \( H^1_0(\Omega) \).

**Theorem 3.** Under the conditions of Theorem 1, we have for sufficiently small \( h \)
\[
|\lambda - \hat{\lambda}_h| \leq C h^{2\min\{k+1, t\}},
\]
which is of higher order than \( |\lambda - \lambda_h| = O(h^{2s}) = O(h^{2\min\{k+1, t\}}) \) if \( t > k+1 \).

Proof. Taking \( \phi = \psi_h \) in (14), we obtain
\[
|\lambda - \hat{\lambda}_h| \leq C \| p - \psi_h \|^2_{1, \Omega}.
\]
The proof is completed by invoking Corollary 1.

**A Posteriori Error Estimates:** We can also use \( \psi_h \) to define the a posteriori error estimates
\[
\eta_u = \| u_h - A \nabla \psi_h \|_{0, \Omega}, \quad \eta_p = \| p_h - \psi_h \|_{0, \Omega}.
\]
Again thanks to the superconvergence results of Corollary 1, the following theorem shows that \( \eta_u \) and \( \eta_p \) are asymptotically exact for the eigenfunction errors \( \| u - u_h \|_{0, \Omega} \) and \( \| p - p_h \|_{0, \Omega} \), respectively, as the mesh size \( h \) tends to zero.
Theorem 4. Under the conditions of Theorem 1, we have for sufficiently small $h$

$$
\left\| \mathbf{u} - \mathbf{u}_h \right\|_{0, \Omega} = \eta_u + O(h^{\min\{k+1+\gamma, t\}}).
$$

Moreover, it holds that

$$
\frac{\eta_u}{\left\| \mathbf{u} - \mathbf{u}_h \right\|_{0, \Omega}} - 1 = O(h^{\min\{\gamma, t-k-1\}}),
$$

provided that $t > k + 1$ and there exists a constant $C > 0$ such that

$$
\left\| \mathbf{u} - \mathbf{u}_h \right\|_{0, \Omega} \geq Ch^{\min\{k+1, t\}} = Ch^k.
$$

Similarly,

$$
\left\| \mathbf{p} - \mathbf{p}_h \right\|_{0, \Omega} = \eta_p + O(h^{\min\{k+1+\gamma, t+\gamma\}})
$$

and

$$
\frac{\eta_p}{\left\| \mathbf{p} - \mathbf{p}_h \right\|_{0, \Omega}} - 1 = O(h^{\min\{\gamma, t+\gamma-k-1\}}),
$$

provided that $t + \gamma > k + 1$ and there exists a constant $C > 0$ such that

$$
\left\| \mathbf{p} - \mathbf{p}_h \right\|_{0, \Omega} \geq Ch^{\min\{k+1, t+\gamma\}} = Ch^{k+1}.
$$

Proof. By Corollary 1 we obtain

$$
\left\| \mathbf{u} - \mathbf{u}_h \right\|_{0, \Omega} - \eta_u \leq \left\| \nabla \mathbf{p} - \nabla \psi_h \right\|_{0, \Omega} \leq C\left\| \mathbf{p} - \psi_h \right\|_{1, \Omega} \leq Ch^{\min\{k+1+\gamma, t\}},
$$

and thus

$$
\frac{\eta_u}{\left\| \mathbf{u} - \mathbf{u}_h \right\|_{0, \Omega}} - 1 \leq C\frac{h^{\min\{k+1+\gamma, t\}}}{h^{k+1}} = Ch^{\min\{\gamma, t-k-1\}}.
$$

The same argument applied to the error $\left\| \mathbf{p} - \mathbf{p}_h \right\|_{0, \Omega}$ gives

$$
\left\| \mathbf{p} - \mathbf{p}_h \right\|_{0, \Omega} - \eta_p \leq \left\| \mathbf{p} - \psi_h \right\|_{0, \Omega} \leq Ch^{\min\{k+1+\gamma, t+\gamma\}},
$$

and thus

$$
\frac{\eta_p}{\left\| \mathbf{p} - \mathbf{p}_h \right\|_{0, \Omega}} - 1 \leq C\frac{h^{\min\{k+1+\gamma, t+\gamma\}}}{h^{k+1}} = Ch^{\min\{\gamma, t+\gamma-k-1\}}.
$$

This completes the proof. \(\square\)

Remark 2. The non-degeneracy conditions (16) and (17) come from the a priori error estimates (5) and (7).
Remark 3. For the lowest-order element \(k = 0\), the a posteriori error estimate \(\| p_h - \Pi_h p_h \|_{0, \Omega} \) given in [11] is much easier to compute and is asymptotically exact for \(\| p - p_h \|_{0, \Omega} \) under the conditions of Theorem 1. But there is only a residual-based estimator for \(\| u - u_h \|_{H(\text{div}, \Omega)} \) which is not asymptotically exact.

4. Numerical Results

In this section we report some numerical results for the Laplace eigenvalue problem

\[
- \Delta p = \lambda p \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial \Omega,
\]

where \(\Omega\) is the unit square \((0, 1)^2\) in the first example and is the L-shaped domain \((-1, 1) \times (-1, 1) \setminus [0, 1] \times [-1, 0]\) in the second example. This problem is solved by the Raviart–Thomas mixed finite element method \((4)\) of the low orders \(k = 0, 1\) with the normalization \(\| p_h \|_{0, \Omega} = 1\). The eigenpairs \((u_h, p_h, \lambda_h)\) are computed by the MATLAB command ‘eigs’.

4.1. Square domain. For the unit square \(\Omega = (0, 1)^2\), it is well known that the eigenvalues of (18) are given by

\[
\lambda_{l,m} = (l^2 + m^2)\pi^2 \quad (l, m = 1, 2, 3, \ldots)
\]

with the associated eigenfunctions

\[
p_{l,m}(x, y) = \sin(l\pi x) \sin(m\pi y).
\]

Note that we have \(\gamma = 1\) in the regularity estimate (6).

As the eigenfunctions are smooth, we consider a sequence of uniform criss-cross meshes constructed by partitioning \(\Omega\) into equal squares of width \(h = 1/2^n\) and then dividing every square into four equal triangles by its two diagonals (see Fig. 1 for \(n = 1, 2, 3\)).

The numerical results are listed in Tables 1–2 and 3–4 for the \(RT_0\) and \(RT_1\) mixed finite element approximations of the first eigenvalue \(\lambda = 2\pi^2\) and its corresponding eigenfunction \(p(x, y) = 2\sin(\pi x) \sin(\pi y)\) with \(\| p \|_{0, \Omega} = 1\). The numerical convergence order and the effectivity index for \(u - u_h\) are computed by

\[
\text{Order} = \log_2 \frac{\| u - u_{2h} \|_{0, \Omega}}{\| u - u_h \|_{0, \Omega}}, \quad I_u = \frac{\eta_u}{\| u - u_h \|_{0, \Omega}};
\]

and similarly for \(p - p_h\) et al., where \(\eta_u\) and \(\eta_p\) are defined by (15).
Figure 1. Uniform criss-cross meshes on $\Omega = (0, 1)^2$ with the mesh size $h = 1/2^n$ ($n = 1, 2, 3$)

It is clearly seen that the convergence orders are in excellent agreement with the theoretical results of the preceding section. Moreover, it is evident that the a posteriori error estimates $\eta_u$ and $\eta_p$ are asymptotically exact for the $L^2$ errors of the eigenfunctions (see Theorem 4) and the improved eigenvalue approximation $\widehat{\lambda}_h$ exhibits the two-order higher superconvergence than $\lambda_h$ (see Theorem 3). It is also worthwhile to observe that the convergence order of $\|p - \psi_h\|_{0,\Omega}$ is not better than that of $|p - \psi_h|_{1,\Omega}$ for the $RT_0$ element, as predicted by Corollary 1, but it is one-order higher for the $RT_1$ element.

| 1/h | $\|u - u_h\|_{0,\Omega}$ | Order $I_u$ | $\|p - p_h\|_{0,\Omega}$ | Order $I_p$ | $|\lambda - \lambda_h|\$ | Order |
|-----|------------------|--------|------------------|--------|------------------|--------|
| 4   | 1.001e+0        | —      | 1.845e-1         | —      | 3.407e-1         | —      |
| 8   | 5.028e-1        | 0.993  | 9.248e-2         | 0.996  | 8.470e-2         | 2.008  |
| 16  | 2.517e-1        | 0.998  | 4.627e-2         | 0.999  | 2.115e-2         | 2.002  |
| 32  | 1.259e-1        | 0.999  | 2.314e-2         | 1.000  | 5.285e-3         | 2.001  |
| 64  | 6.296e-2        | 1.000  | 1.157e-2         | 1.000  | 1.321e-3         | 2.000  |

Table 1. Errors and effectivity indices for the $RT_0$ eigenpair $(u_h, p_h, \lambda_h)$ on $\Omega = (0, 1)^2$

4.2. L-shaped domain. For $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1] \times [-1, 0]$, it is known that the eigenfunction $(u, p)$ corresponding to the first eigenvalue $\lambda$ has a singularity at the origin and belongs to the space $H^t(\Omega)^2 \times H^{1+t}(\Omega)$ with $t = \frac{2}{3} - \varepsilon$ for any $\varepsilon > 0$.

To approximate the first eigenpairs $(u, p, \lambda)$, we successively apply adaptive mesh refinement based on the longest-edge bisection starting
Table 2. Errors for the $P_2$-postprocessed eigenpair $(\psi_h, \hat{\lambda}_h)$ on $\Omega = (0,1)^2$

| $1/h$ | $|p - \psi_h|_{1,\Omega}$ Order | $\|p - \psi_h\|_{0,\Omega}$ Order | $\lambda_h - \lambda$ Order |
|-------|-------------------------------|-------------------------------|----------------------------|
| 4     | 1.819e-1                      | 3.420e-2                     | 1.071e-2                  |
| 8     | 4.531e-2                      | 8.560e-3                     | 6.167e-4                  |
| 16    | 1.131e-2                      | 2.141e-3                     | 3.766e-5                  |
| 32    | 2.828e-3                      | 5.354e-4                     | 2.339e-6                  |
| 64    | 7.069e-4                      | 1.339e-4                     | 1.460e-7                  |

Table 3. Errors and effectivity indices for the $RT_1$ eigenpair $(u_h, p_h, \lambda_h)$ on $\Omega = (0,1)^2$

| $1/h$ | $\|u - u_h\|_{0,\Omega}$ Order | $I_u$ | $\|p - p_h\|_{0,\Omega}$ Order | $I_p$ | $|\lambda - \lambda_h|$ Order |
|-------|-------------------------------|-------|-------------------------------|-------|-------------------------------|
| 2     | 3.023e-1                      | 0.9714| 7.028e-2                      | 0.9445| 6.145e-3                      |
| 4     | 7.499e-2                      | 0.9927| 1.773e-2                      | 0.9856| 5.803e-4                      |
| 8     | 1.874e-2                      | 0.9982| 4.447e-3                      | 0.9963| 3.897e-5                      |
| 16    | 4.686e-3                      | 0.9995| 1.113e-3                      | 0.9991| 2.477e-6                      |
| 32    | 1.172e-3                      | 0.9999| 2.782e-4                      | 0.9998| 1.554e-7                      |

Table 4. Errors for the $P_3$-postprocessed eigenpair $(\psi_h, \hat{\lambda}_h)$ on $\Omega = (0,1)^2$

| $1/h$ | $|p - \psi_h|_{1,\Omega}$ Order | $\|p - \psi_h\|_{0,\Omega}$ Order | $\lambda_h - \lambda$ Order |
|-------|-------------------------------|-------------------------------|----------------------------|
| 2     | 7.690e-2                      | 5.365e-3                     | 5.369e-3                  |
| 4     | 9.892e-3                      | 3.268e-4                     | 9.577e-5                  |
| 8     | 1.254e-3                      | 2.051e-5                     | 1.563e-6                  |
| 16    | 1.573e-4                      | 1.283e-6                     | 2.471e-8                  |
| 32    | 1.968e-5                      | 8.024e-8                     | 3.913e-10                 |

with the initial mesh displayed in the left of Fig. 2. In light of (15), the local error estimator is defined as

$$\eta_K^2 = \|u_h - \nabla \psi_h\|_{0,K}^2 + \|p_h - \psi_h\|_{0,K}^2 \quad \forall K \in T_h,$$
and the element $K$ is marked for refinement if $\eta_K \geq \frac{1}{2} \max_{K' \in \mathcal{T}_h} \eta_{K'}$. Then more elements of $\mathcal{T}_h$ are marked for refinement to avoid hanging nodes.

Fig. 2 displays some adaptively refined meshes generated by the above process. We see that the local error estimator $\eta_K$ is able to detect the singularity of the eigenfunction around which the mesh refinement is concentrated.

Fig. 3 plots the errors for the eigenvalue approximation $\lambda_h$ and the postprocessed eigenvalue approximation $\hat{\lambda}_h$ in terms of the number of unknowns $N$. Since the exact value of $\lambda$ is not known, we take the numerical value $\lambda \approx 9.6397238440219$ used in [11] as the exact one. It is observed that $\hat{\lambda}_h$ exhibits the almost one-order higher superconvergence than $\lambda_h$ and the errors decay at nearly optimal orders, where the convergence order is computed in powers of $1/N$ for adaptively refined meshes.

To check the asymptotic exactness of the a posteriori error estimates $\eta_u$ and $\eta_p$, we use the following equality (cf. Lemma 4 of [8])

$$\lambda - \lambda_h = \| u - u_h \|_{0, \Omega}^2 - \lambda_h \| p - p_h \|_{0, \Omega}^2,$$

which holds for the Laplace eigenvalue problem (18) with $\| p \|_{0, \Omega} = \| p_h \|_{0, \Omega} = 1$. This leads us to define the effectivity index for the eigenvalue error by

$$I_\lambda = \frac{\eta_u^2 - \lambda_h \eta_p^2}{\lambda - \lambda_h},$$

The results are plotted in Fig. 4 which shows that $I_\lambda$ eventually converges to 1. This indicates that $\eta_u$ and $\eta_p$ are asymptotically exact for the $L^2$ errors of the eigenfunctions.

![Figure 2](image1.png)

**Figure 2.** Initial (left) and adaptively refined meshes for $RT_0$ (middle) and $RT_1$ (right)
Figure 3. Eigenvalue errors for $RT_0$ (left) and $RT_1$ (right)

Figure 4. Effectivity indices for $RT_0$ (left) and $RT_1$ (right)

References


Postprocessing for the R-T mixed finite element for eigenvalue problem


Kwang-Yeon Kim
Department of Mathematics
Kangwon National University
Chun-Cheon 24341, Korea
E-mail: eulerkim@kangwon.ac.kr